The supremum of Newton polygons of p-divisible groups with a given p-kernel type

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Abstract

In this paper we show that there exists the supremum of Newton polygons of p-divisible groups with a given p-kernel type, and provide an algorithm determining it.

1 Introduction

We are concerned with estimating the isogeny type (=Newton polygon, cf. [10]) of a p-divisible group X from its p-kernel $X[p]$. In this paper we give an optimal estimation.

We fix once for all, non-negative integers c and d with $r := c + d > 0$. Let $\mathrm{W}(= \mathrm{W}_r)$ be the Weyl group of the general linear group GL_r. In the usual sense, we identify W and $\mathrm{Aut}(\{1, \dots, r\})$. Let $s_i \in \mathrm{W}$ be the simple reflection $(i, i + 1)$ for $i = 1, \dots, r - 1$. Let $\mathrm{S} = \{s_1, \dots, s_{r-1}\}$ and set $\mathrm{J} := \mathrm{S} \setminus \{s_d\}$. Let $\mathrm{W_J}$ be the subgroup of W generated by elements of J. We denote by ${}^{\mathrm{J}}\mathrm{W}$ the set of $(\mathrm{J}, \emptyset)$-reduced elements of W (cf. [1], Chap. IV, Ex. §1, 3), which are representatives of $\mathrm{W_J} \setminus \mathrm{W}$. A classification theory of BT_1's by Kraft, Oort, Moonen and Wedhorn says that the set of the isomorphism classes of BT_1's with tangent-dimension d and length r is bijective to the set ${}^{\mathrm{J}}\mathrm{W}$. Note that ${}^{\mathrm{J}}\mathrm{W}$ has a natural ordering $\subset$ introduced and investigated by Wedhorn [21] (see §5 for a short review).

Let w be any element of ${}^{\mathrm{J}}\mathrm{W}$. In Corollary 2.2, we show that there exists the supremum $\xi(w)$ of Newton polygons of p-divisible groups with p-kernel type w:

- every p-divisible group whose p-kernel is of type w has Newton polygon $\prec \xi(w)$;
- there exists a p-divisible group X such that $X[p]$ is of type w and the Newton polygon of X equals $\xi(w)$.

The following theorem gives us a combinatorial algorithm determining $\xi(w)$, see Remark 7.1. Let $\mu(\zeta) \in {}^{\mathrm{J}}\mathrm{W}$ denote the type of the p-kernel of the minimal p-divisible group $H(\zeta)$ (cf. [14] and also a review [4], §3).

Theorem 1.1. *$\xi(w)$ is the biggest one of Newton polygons ζ such that $\mu(\zeta) \subset w$.*

This is an unpolarized analogue of [6], Corollary II. For a more effective algorithm determining the first/last slope of $\xi(w)$, see [3], Theorem 4.1 for the polarized case and [4], Corollary 1.3 for the unpolarized case. Recall that in the polarized case the existence of the supremum $\xi(w)$ follows from the irreducibility of Ekedahl-Oort strata on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties ([2], Theorem 11.5). An obstruction in the unpolarized case has been the absence of a good moduli space like $\mathcal{A}_g$. However using Vasiu's $\mathbb{T}_m$-action instead, we have (Lemma 2.1) that there exists an irreducible catalogue of p-divisible groups with a fixed p-kernel type; this clearly shows the existence of $\xi(w)$. Then Theorem 1.1 can be shown by a similar argument as in [6] (which is relatively easier than the polarized case). Finally we mention a different approach announced by Viehmann [20], who seems to have generalized our results in terms of the loop groups of split reductive groups, making use of results on affine Deligne-Lusztig varieties.

Terminology

We naturally identify the category of affine schemes with the opposite category to the category of commutative rings. We fix once for all a rational prime p. In this paper we freely uses a part of Zink's result [22], Theorem 9, which says that for a commutative ring R of finite type over a field of characteristic p, there exists a categorical equivalence from the category of p-divisible groups over R to that of nilpotent displays over R, where we follow the terminology of [9] for displays and nilpotent displays.

2 A catalogue of p-divisible groups with a given type

Let k be an algebraically closed field of characteristic p. Let $(P, Q, F, \dot{F})$ be a display over k, and $P = L \oplus T$ be a normal decomposition ([22], Introduction). Let $\mathrm{G} = \mathrm{GL}(P)$ be the general linear group over $W(k)$ of degree $r = c + d$. Let H be the paraholic subgroup of G stabilizing Q, which is a connected smooth affine group scheme over $W(k)$. Let $\mathcal{D}_m$ and $\mathcal{H}_m$ be connected smooth affine group schemes over k such that $\mathcal{D}_m(R) = \mathrm{G}(W_m(R))$ and $\mathcal{H}_m(R) = \mathrm{H}(W_m(R))$ respectively, see [18], 2.1.4 for more details. For any truncated Barsotti-Tate groups of level m with codimension c and dimension d, its Dieudonné module is written as $(P/p^m P, gF, Vg^{-1})$ for some $g \in \mathcal{D}_m$. Vasiu introduced an action:

$$\mathbb{T}_m : \quad \mathcal{H}_m \times_k \mathcal{D}_m \longrightarrow \mathcal{D}_m, \tag{1}$$

and showed in [18], 2.2.2 that the set of $\mathbb{T}_m$-orbits are naturally bijective to the set of isomorphism classes of truncated Barsotti-Tate groups of level m over k with codimension c and dimension d. Let $\mathbf{BT}_m(k)$ be the set of isomorphism classes of truncated Barsotti-Tate groups of level m over k with codimension c and dimension d. We have

Lemma 2.1. *For any $u \in \mathbf{BT}_m(k)$, there exists an irreducible catalogue of p-divisible groups with p^m-kernel type u, i.e., there exists a family $\mathcal{X} \to S$ of p-divisible groups such that*

(1) *for any geometric point $s \in S$, the p^m-kernel of the fiber $\mathcal{X}_s$ is of type u;*

(2) *For any p-divisible group X with p^m-kernel type u, there exists a geometric point $s \in S$ such that $X \simeq \mathcal{X}_s$;*

(3) *S is irreducible and of finite type over k.*

Proof. It suffices to show the case that u has no étale part, since every (truncated) Barsotti-Tate group over k is the direct sum of its local part and its étale part and the decomposition is compatible with truncations. Let N be an integer $\geq m$ so that $X[p^N] \simeq Y[p^N]$ implies $X \simeq Y$ for any p-divisible groups X and Y over k (cf. [15], 1.7 and [19]). Let π be the natural map $\mathcal{D}_N \to \mathcal{D}_m$, and let τ be a section of $\mathcal{D} \to \mathcal{D}_N$. Let $\mathbb{O}_u$ be the $\mathbb{T}_m$-orbit associated to u. Since $\mathcal{H}_m$ is irreducible, $\mathbb{O}_u$ is irreducible. Since π is smooth with connected fibers, $\pi^{-1}(\mathbb{O}_u)$ is also irreducible. Let S be the image of $\pi^{-1}(\mathbb{O}_u)$ by τ. Then S is irreducible and of finite type over k. By [22], Theorem 9, we have a p-divisible groups $\mathcal{X}$ over S. Clearly $\mathcal{X}$ satisfies the required properties. □

Corollary 2.2. *There exists the supremum of Newton polygons of p-divisible groups with a given p^m-kernel type.*

Proof. Let $\mathcal{X} \to S$ be the family as in the lemma above. Let η be the generic point of S. It follows from Grothendieck and Katz ([7], Th. 2.3.1 on p. 143) that the Newton polygon of $\mathcal{X}_\eta$ is the supremum of Newton polygons of p-divisible groups with a given p^m-kernel type. $\square$

3 F-zips

Let S be a scheme of characteristic p. Let σ denote the absolute Frobenius on S. For any $\mathcal{O}_S$-module M we write $M^{(p)} = \mathcal{O}_S \otimes_{\sigma,\mathcal{O}_S} M$. Recall the definition ([12], (1.5)) of F-zip in a particular case.

Definition 3.1. An *F-zip* over S is a quintuple $Z = (N, C, D, \varphi, \dot{\varphi})$ consisting of locally free $\mathcal{O}_S$-module N and $\mathcal{O}_S$-submodules C, D of N which are locally direct summands of N, and σ-linear homomorphisms $\varphi : N/C \to D$ and $\dot{\varphi} : C \to N/D$ whose $\mathcal{O}_S$-linearizations $\varphi^\sharp : (N/C)^{(p)} \to D$ and $\dot{\varphi}^\sharp : C^{(p)} \to N/D$ are isomorphisms. If S is connected, we define the *height* of Z to be the rank of N and the *type* of Z to be a map from $\{0, 1\}$ to $\mathbb{Z}_{\geq 0}$ sending 0 to $\operatorname{rk} D$ and 1 to $\operatorname{rk} C$; we will simply write the type as $(\operatorname{rk} D, \operatorname{rk} C)$.

If S is the spectrum of a perfect field K, then the (covariant) Dieudonné functor $\mathbb{D}$ makes an equivalence from the category of BT_1's over K to that of F-zips over K. The F-zip $(N, C, D, \varphi, \dot{\varphi})$ associated to a BT_1-group G is given by $N = \mathbb{D}(G)$ with $C = VN$ and $D = FN$, and φ and $\dot{\varphi}^{-1}$ are naturally induced by F and V respectively.

As shown by Kraft, Oort, Moonen and Wedhorn, there exists a bijection from $\mathbf{BT}_1(k)$ to ${}^{\mathrm{J}}\mathrm{W}$ for any algebraically closed field k of characteristic p (this statement is due to [12]; also see [8], [11] and [16]). This classification is based on the fact that for any BT_1-group G over k, there uniquely exists $w \in {}^{\mathrm{J}}\mathrm{W}$ such that G is isomorphic to G_w defined below. The notion of final type is useful. To a $w \in {}^{\mathrm{J}}\mathrm{W}$ we associate a pair (B, δ), called a *final type*, cf. [4], Definition 2.6, where B is a totally ordered set $\{b_1 < \ldots < b_r\}$ and δ is a map $B \to \{0, 1\}$ defined by $\delta(b_i) = 1 \Leftrightarrow w(i) \leq d$. There uniquely exists an automorphism $\pi = \pi_\delta$ of B such that $\pi(b') > \pi(b) \Leftrightarrow \delta(b') > \delta(b)$ for any $b' < b$. We define G_w so that its F-zip $Z_w = (N, C, D, \varphi, \dot{\varphi})$ is given by $N = \sum_{b \in B} kb$ (i.e., the k-vector space with basis indexed by B) and $C = \sum_{\delta(b)=1} kb$, and $D = \sum_{\delta(b)=0} k\pi(b)$ and φ, $\dot{\varphi}$ are defined by $\varphi(b) = \pi(b)$ for b with $\delta(b) = 0$ and $\dot{\varphi}(b) = \pi(b)$ for b with $\delta(b) = 1$.

4 Homomorphisms of F-zips

Let k be an algebraically closed field of characteristic p. In this section every scheme will be over k. We first review a description of homomorphisms of F-zips for the reader's convenience (cf. [13], §2 and [11], §4 and also see [4], § 4.3), and show some facts used later on.

Let w_1 and w_2 be the types of Z_1 and Z_2 respectively. Let $\mathcal{B}_1 = (B_1, \delta_1)$ and $\mathcal{B}_2 = (B_2, \delta_2)$ be their final types and set $\pi_1 = \pi_{\delta_1}$ and $\pi_2 = \pi_{\delta_2}$. A *finite slice* ω is a subset of $B_1 \times B_2$ of the form $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ with $|\omega| = \ell$ for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying **(a)** $\delta_1(s_1) = 1$ and $\delta_2(s_2) = 0$, **(b)** $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \leq i < \ell$ and **(c)** $\delta_1(\pi_1^\ell(s_1)) = 0$ and $\delta_2(\pi_2^\ell(s_2)) = 1$. We denote by $\Omega_o = \Omega_o(\mathcal{B}_1, \mathcal{B}_2)$ the set of finite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$. An *infinite slice* ω is a subset of $B_1 \times B_2$ of the form $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ with $|\omega| = \ell$ for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying **(a)** $s_1 = \pi_1^\ell(s_1)$ and $s_2 = \pi_2^\ell(s_2)$, **(b)** $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \leq i < \ell$. We denote by $\Omega_\infty = \Omega_\infty(\mathcal{B}_1, \mathcal{B}_2)$ the set of infinite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$. Set $\Omega = \Omega(\mathcal{B}_1, \mathcal{B}_2) := \Omega_o \sqcup \Omega_\infty$. For each slice ω, we define a group scheme $\mathbb{K}_\omega$ to be the additive group $\mathbb{G}_a$ if $\omega \in \Omega_o$ and to be $\operatorname{Ker}(F^{|\omega|} - \mathrm{id} : \mathbb{G}_a \to \mathbb{G}_a)$ if $\omega \in \Omega_\infty$. Let S be a k-scheme. Let $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ be a slice with $|\omega| = \ell$. For an element $r \in \omega$, we denote by $\varepsilon(r)$ $(= \varepsilon_\omega(r))$ the integer ε with $0 \leq \varepsilon < \ell$ satisfying $r = (\pi_1^{\varepsilon+1}(s_1), \pi_2^{\varepsilon+1}(s_2))$. For $a \in \mathbb{K}_\omega(S)$, we define a map

$$\mathrm{st}_{\omega,a} : \quad B_1 \times B_2 \longrightarrow \mathbb{K}_\omega(S) \tag{2}$$

by sending $r \in \omega$ to $a^{p^{\varepsilon(r)}}$ and $r \notin \omega$ to 0. The functor, from the category of k-schemes to the category of commutative groups, sending S to $Hom_S(Z_{1,S}, Z_{2,S})$ is represented by a group scheme $\mathrm{Hom}(Z_1, Z_2)$; moreover there is an isomorphism as group schemes:

$$\Lambda : \quad \bigoplus_{\omega \in \Omega} \mathbb{K}_\omega \xrightarrow{\sim} \mathrm{Hom}(Z_1, Z_2). \tag{3}$$

Indeed we write $B_* = \{b_1^{(*)} < \cdots < b_{r_*}^{(*)}\}$ and also write $Z_* = (N_*, C_*, D_*, \varphi_*, \dot{\varphi}_*)$ with $N_* = \bigoplus_{i=1}^{r_*} k b_i^{(*)}$. Let S be any k-scheme. An $\mathcal{O}_S$-homomorphism $\mu : N_{1,S} \to N_{2,S}$, say $\mu(b_i^{(1)}) = \sum_j r_{ij} b_j^{(2)}$ with $r_{ij} \in \Gamma(S, \mathcal{O}_S)$ gives an element of $Hom_S(Z_{1,S}, Z_{2,S})$ if and only if r_{ij} is of the form $\sum_{\omega\in\Omega} \mathrm{st}_{\omega,a}(b_{ij})$ for a certain $a \in \mathbb{K}_\omega(S)$, where $b_{ij} = (b_i^{(1)}, b_j^{(2)}) \in B_1 \times B_2$. From now on we identify $\mathrm{Hom}(Z_1, Z_2)$ with $\bigoplus_{\omega\in\Omega} \mathbb{K}_\omega$. The connected component of zero in a commutative group scheme G will be denoted by G_o. Then $\mathrm{Hom}(Z_1, Z_2)_o$ is the product of $\mathbb{K}_\omega$ for $\omega \in \Omega_o$. We write $\mathrm{Hom}(Z_1, Z_2)_\infty$ for $\bigoplus_{\omega\in\Omega_\infty} \mathbb{K}_\omega$.

It is straightforward to prove

Lemma 4.1. *Let Z_1, Z_2, Z_3 be F-zips over k. The composition map*

$$\mathrm{Hom}(Z_1, Z_2) \times \mathrm{Hom}(Z_2, Z_3) \longrightarrow \mathrm{Hom}(Z_1, Z_3)$$

sends the pair of (ω_1, a_1) and (ω_2, a_2) (i.e., $\omega_i \in \Omega(\mathcal{B}_i, \mathcal{B}_{i+1})$ and $a_i \in \mathbb{K}_{\omega_i}$ for $i = 1, 2$) to $\sum_{\tilde{\omega}} (\omega, a_1^{p^e} a_2^{p^f})$, where the sum is over $\pi_1 \times \pi_2 \times \pi_3$-orbits $\tilde{\omega}$ in $\omega_1 \times_{B_2} \omega_2$ and $\omega = \mathrm{pr}_{13}(\tilde{\omega})$ and e is the minimal element of $\varepsilon_{\omega_1}(\mathrm{pr}_{12}(\tilde{\omega}))$ and f is the minimal element of $\varepsilon_{\omega_2}(\mathrm{pr}_{23}(\tilde{\omega}))$. Here we denote by pr_{ij} the projections $B_1 \times B_2 \times B_3 \to B_i \times B_j$ for $1 \le i < j \le 3$.

The next lemma shows that the ring scheme $\mathrm{End}(Z)_o$ consists of nilpotent endomorphisms.

Lemma 4.2. *Let $\omega \in \Omega_o(\mathcal{B}, \mathcal{B})$. Let (b, b') be an element of ω. Then we have $b > b'$.*

Proof. By the definition of finite slice, $\nu(b) := \sum_{l\in\mathbb{N}} \delta(\pi^{-l}(b)) 2^{-l}$ is greater than $\nu(b') := \sum_{l\in\mathbb{N}} \delta(\pi^{-l}(b')) 2^{-l}$. Then [5], Proposition 4.7 shows $b > b'$. □

Finally we look at the action of $\mathrm{End}(Z)_o$ on $\mathrm{Hom}(Z, Z_1)$. Let $\Omega^i(\mathcal{B}, \mathcal{B}_1)$ be the subset of $\Omega(\mathcal{B}, \mathcal{B}_1)$ consisting of $\omega \in \Omega(\mathcal{B}, \mathcal{B}_1)$ with $\mathrm{pr}(\omega) \subset \{b_i, \ldots, b_r\}$, where pr is the projection $B \times B_1 \to B$. We define a subgroup scheme $\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1)$ of $\mathrm{Hom}(Z, Z_1)$ by

$$\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) = \bigoplus_{\omega \in \Omega^i(\mathcal{B}, \mathcal{B}_1)} \mathbb{K}_\omega. \tag{4}$$

From the lemmas above, we have

Corollary 4.3. *The composition map induces*

$$\mathrm{End}(Z)_o \times \mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) \longrightarrow \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1).$$

5 Specialization of F-zips

We recall Wedhorn's results in [21].

Definition 5.1. Let w and w' be elements of ${}^{\mathrm{J}}\mathrm{W}$. We say $w \subset w'$ if there exists a family $Z \to S$ of F-zips over an irreducible scheme S such that the isomorphism type of a special fiber is w and that of the generic fiber is w'.

Let $x = w_0^{\mathrm{J}} : \mathrm{W} \to \mathrm{W}$ be the map sending i to $i + c$ if $i \le d$ and i to $i - d$ if $i > d$. Define $\delta : \mathrm{W} \to \mathrm{W}$ by $\delta(u) = x \cdot u \cdot x^{-1}$.

Theorem 5.2 ([21])**.** *Let $w, w' \in {}^{\mathrm{J}}\mathrm{W}$. We have $w \subset w'$ if and only if there exists $u \in \mathrm{W}_{\mathrm{J}}$ such that $u^{-1} w \delta(u)$ is less than or equal to w' with respect to the Bruhat order.*

6 Lifting of F-zips

Let R be a commutative ring of characteristic p. Let ^{F} and ^{V} denote the Frobenius and Verschiebung on $W(R)$. Write $I_R := {}^V W(R)$. Let $M = (P, Q, F, \dot{F})$ be a display over R. One can associate to M an F-zip $M/I_R M$, which is defined as follows. Let $P = L \oplus T$ be a normal decomposition of P with $Q = L \oplus I_R T$ (cf. [22], Introduction). Write $M/I_R M = (N, C, D, \varphi, \dot{\varphi})$. Then we define $N = P/I_R P$ and $C = Q/I_R P \simeq L/I_R L$, and D is the submodule of N generated by the image of $F : T \to P \to N$, and φ and $\dot{\varphi}$ are canonically induced by F and $\dot{F}$ respectively.

Lemma 6.1. *Let Z be an F-zip over S. Let s be any closed point of S. Let M be a display over s. There is an open affine subscheme $U = \operatorname{Spec}(R)$ of S with $s \in U$ and a display $\mathcal{M}$ over R such that $\mathcal{M}/I_R\mathcal{M} \simeq Z_R$ and $\mathcal{M}_s \simeq M$.*

Proof. Write $Z = (N, C, D, \varphi, \dot{\varphi})$. Let U be an affine open subscheme of S containing s over which C and D are direct summands of N, say $N = C \oplus E$, and C, D and E are free. Write $U = \operatorname{Spec}(R)$ and $s = \operatorname{Spec}(R/\mathfrak{m})$. We replace S by U. We have an ^{F}-linear homomorphism

$$\phi : \quad C \oplus E \xrightarrow{\ \sim\ } C \oplus N/C \xrightarrow{\dot{\varphi} \oplus \varphi} N/D \oplus D \xrightarrow{\ \sim\ } N.$$

Let $\mathcal{L}$ and $\mathcal{T}$ be free $W(R)$-modules such that $\mathcal{L}/I_R\mathcal{L} \simeq C$ and $\mathcal{T}/I_R\mathcal{T} \simeq E$. Put $\mathcal{P} = \mathcal{L} \oplus \mathcal{T}$ and $\mathcal{Q} = \mathcal{L} \oplus I_R\mathcal{T}$. Let $M = (P, Q, F, \dot{F})$ and $L \oplus T$ a normal decomposition of P, and let Φ_0 be $\dot{F} \oplus F : L \oplus T \to P$ obtained in [22], Lemma 9; one can identify L and T with $\mathcal{L}/W(\mathfrak{m})\mathcal{L}$ and $\mathcal{T}/W(\mathfrak{m})\mathcal{T}$ respectively.

Since the canonical map from $\mathrm{GL}_r(W(R))$ to the fiber product of $\mathrm{GL}_r(R) \to \mathrm{GL}_r(R/\mathfrak{m})$ and $\mathrm{GL}_r(W(R/\mathfrak{m})) \to \mathrm{GL}_r(R/\mathfrak{m})$ is clearly surjective, there exists an ^{F}-linear homomorphism

$$\Phi : \quad \mathcal{L} \oplus \mathcal{T} \longrightarrow \mathcal{P}$$

such that $(\Phi \mod I_R) = \phi$ and $(\Phi \mod W(\mathfrak{m})) = \Phi_0$. Set $\mathcal{F} = \Phi \circ ({}^V 1 \oplus \mathrm{id}) : \mathcal{L} \oplus \mathcal{T} \to \mathcal{P}$ and define $\dot{\mathcal{F}} : \mathcal{L} \oplus I_R\mathcal{T} \to \mathcal{P}$ by sending $l + {}^V wt$ to $\Phi(l) + w\Phi(t)$ for every $l \in \mathcal{L}$, $t \in \mathcal{T}$ and $w \in W(R)$. Then we have a display $\mathcal{M} = (\mathcal{P}, \mathcal{Q}, \mathcal{F}, \dot{\mathcal{F}})$, which satisfies the required properties. □

Corollary 6.2. *Let w and w' be elements of ${}^{\mathrm{J}}\mathrm{W}$. If $w \subset w'$, then we have $\xi(w) \prec \xi(w')$.*

Proof. Assume $w \subset w'$, i.e., there exists an F-zip Z over an irreducible scheme S such that the type of the fiber of the generic point η is w' and the type of the fiber of a special point s is w. By Corollary 2.2, there exists a display M over s such that the Newton polygon of M is $\xi(w)$. Applying Lemma 6.1 to Z and M, there exist an open affine subscheme $U = \operatorname{Spec}(R)$ of S containing s and a display $\mathcal{M}$ over R such that $\mathcal{M}/I_R\mathcal{M} \simeq Z_R$ and $\mathcal{M}_s \simeq M$. It follows from Grothendieck-Katz ([7], Th. 2.3.1 on p. 143) that $\xi(w)$ is less than or equal to the Newton polygon, say ζ, of $\mathcal{M}_\eta$. By the definition of $\xi(w')$, we have $\zeta \prec \xi(w')$. □

7 A reduction of the problem

Let w be any element of ${}^{\mathrm{J}}\mathrm{W}$. Let $\xi(w)$ be the Newton polygon introduced in §1. Its existence is showed in Corollary 2.2. The purpose of this paper is to prove Theorem 1.1:

$$\xi(w) = \max_{\prec}\{\zeta \mid \mu(\zeta) \subset w\}, \tag{5}$$

where ζ is over Newton polygons $\sum(m_i, n_i)$ with $\sum m_i = d$ and $\sum n_i = c$.

Remark 7.1. This gives, thanks to Theorem 5.2, a purely combinatorial algorithm determining $\xi(w)$ for a given w. See [5], Corollary 4.8 for a way to compute $\mu(\zeta)$.

We first prove that Theorem 1.1 follows from the next proposition. The subsequent sections are devoted to the proof of this proposition.

Proposition 7.2. *Assume that w is not minimal. Then there exists a non-constant family of isogenies of p-divisible groups*

$$H(\xi(w))_S \longrightarrow \mathcal{X} \tag{6}$$

over S such that the isomorphism type of $\mathcal{X}_s[p]$ is w for every geometric point s of S.

Proof of (Proposition 7.2 $\Rightarrow$ Theorem 1.1). We first claim that Theorem 1.1 is equivalent to

$$\mu(\xi(w)) \subset w. \tag{7}$$

Clearly Theorem 1.1 implies (7). Suppose (7). Put $\Xi = \{\zeta \mid \mu(\zeta) \subset w\}$. We want to show that $\xi(w)$ is the biggest element of Ξ. Clearly (7) says $\xi(w) \in \Xi$. Let ζ be any element of Ξ. Then we have $\xi(\mu(\zeta)) \prec \xi(w)$ by Corollary 6.2. Note that we have $\xi(\mu(\zeta)) = \zeta$ by [14], (1.2) Theorem. Thus we have $\zeta \prec \xi(w)$.

Let us prove (7) under the assumption that Proposition 7.2 holds. We first consider the case that w is minimal, say $w = \mu(\zeta)$ the type of $H(\zeta)[p]$. Then we have $\xi(w) = \zeta$ by [14] (1.2), and therefore we have $\mu(\xi(w)) = w$; hence (7) holds in this case. Assume that w is not minimal. Let $\mathcal{M}$ be the moduli space of quasi-isogenies $H(\xi(w)) \to Y$ of p-divisible groups, see [17], Chapter 2. Let $\mathcal{I}$ be an irreducible component of $\mathcal{M}_{\mathrm{red}}$ containing the generic point of the family (6). Note that $\mathcal{I}$ is projective ([17], Proposition 2.32). Let $\mathcal{S}_w(\mathcal{I})$ be the locally closed subvariety consisting of isogenies $H(\xi(w)) \to X$ where $X[p]$ is of type w. It is known that $\mathcal{S}_w(\mathcal{I})$ is quasi-affine (cf. [18], 1.2 (g)). By the assumption, we have $\dim \mathcal{S}_w(\mathcal{I}) \geq 1$. Hence there exists $w' \in {}^{\mathrm{J}}\mathrm{W}$ such that $w' \subsetneq w$ and $\xi(w') = \xi(w)$. This shows in particular that the final type v with $\xi(v) = \xi(w)$ which is "minimal w.r.t. $\subset$" is minimal. We use induction on w with respect to $\subset$. Now we assume $\mu(\xi(w')) \subset w'$. Then we have $\mu(\xi(w)) = \mu(\xi(w')) \subset w' \subset w$. $\square$

8 Extensions by a minimal p-divisible group

Let ξ be a Newton polygon. Let $\varrho = (m_1, n_1)$ be a segment of ξ, i.e., $m_1/(m_1 + n_1)$ is a slope of ξ and $\gcd(m_1, n_1) = 1$. Let ξ' be the Newton polygon such that $\xi = \xi' + \varrho$. Let $Z_1 = Z(\varrho)_S$, where $Z(\varrho)$ is the F-zip of $H(\varrho)[p]$ over k. Let

$$0 \longrightarrow Z' \longrightarrow Z \xrightarrow{\ f\ } Z_1 \longrightarrow 0 \tag{8}$$

be a short exact sequence of F-zips over a reduced k-scheme S. Write $Z = (N, C, D, \varphi, \dot\varphi)$ and $Z_1 = (N_1, C_1, D_1, \varphi_1, \dot\varphi_1)$ and so on. That f is surjective means that $f : N \to N_1$ and $f : C \to C_1$ are surjective, and also the injectivity is the dual notion of this surjectivity. Let $M' = (P', Q', F', \dot F')$ be any display lifting Z' with an isogeny $\rho' : M(\xi')_S \to M'$, where $M(\xi')$ is the the display of $H(\xi')$. Let $P' = L' \oplus T'$ be a normal decomposition.

Proposition 8.1. *For any closed point $s \in S$, there exist an open affine subscheme U of S with $s \in U$ and a finite surjective morphism $\mathrm{Spec}(R) \to U$ such that there exist a display $\mathcal{M}$ over R with an isogeny $\rho : M(\xi)_R \to \mathcal{M}$ and a surjective homomorphism $\phi : \mathcal{M} \to (M_1)_R := M(\varrho)_R$ with kernel M'_R and an isomorphism $\theta : \mathcal{M}/I_R\mathcal{M} \to Z_R$ such that we have the commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\xi')_R & \longrightarrow & M(\xi)_R & \xrightarrow{\mathrm{pr}} & M(\varrho)_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \| & & \\ 0 & \longrightarrow & M'_R & \longrightarrow & \mathcal{M} & \xrightarrow{\ \phi\ } & (M_1)_R & \longrightarrow & 0 \end{array} \tag{9}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_R/I_RM'_R & \longrightarrow & \mathcal{M}/I_R\mathcal{M} & \xrightarrow{\overline{\phi}} & (M_1)_R/I_R(M_1)_R & \longrightarrow & 0 \\ & & \| & & \theta\downarrow\simeq & & \| & & \\ 0 & \longrightarrow & Z'_R & \longrightarrow & Z_R & \xrightarrow{f_R} & (Z_1)_R & \longrightarrow & 0. \end{array} \tag{10}$$

Proof. Let $u = \min\{m_1, n_1\}$. Recall [4], Lemma 3.3 that the Dieudonné module $M(\varrho)$ is generated over the Dieudonné ring by X_i ($i \in \mathbb{Z}/u\mathbb{Z}$) and all relations are generated by $F^{\alpha_i}X_i - V^{\beta_{i+1}}X_{i+1} = 0$ for some non-negative integers α_i, β_i. Put $x_i := V^{\beta_i}X_i$.

Let s be any closed point of S. Let $U = \mathrm{Spec}(R)$ be an affine open subscheme of S containing s. We may replace S by U. We choose a lift $\overline{Y}_i \in N$ of $\overline{X}_i \in N_1$ for each $i \in \mathbb{Z}/u\mathbb{Z}$. Let ψ be the composition of $N \to N/D$ and $(\dot{\varphi}^\sharp)^{-1} : N/D \to C^{(p)}$. After replacing R by its finite cover, we can find $\overline{y}_{i,j} \in C$ lifting $V^j\overline{X}_i$ for $0 \le j \le \beta_i$ such that the composition $\psi^{(p^{j-1})} \circ \cdots \circ \psi^{(p)} \circ \psi$ sends $\overline{Y}_i$ to $1 \otimes \overline{y}_{i,j} \in R \otimes_{F^j, R} C$. We put $\overline{y}_i := \overline{y}_{i,\beta_i}$. After replacing R by its open affine subscheme, we can find a section of $N \to N/D$, defining a lift $\tilde{\varphi} : C \to N$ of $\dot{\varphi} : C \to N/D$, such that $\tilde{\varphi}^{\beta_i - j}(\overline{y}_i) = \overline{y}_{i,j}$. It follows from the exact sequence (8) that N is generated by elements of N' and $\tilde{\varphi}^s\overline{y}_i$ ($0 \le s < \beta_i$) and $\varphi^r\tilde{\varphi}^{\beta_i}\overline{y}_i$ ($0 \le r < \alpha_i$) with relations

$$\varphi^{\alpha_i}\tilde{\varphi}^{\beta_i}\overline{y}_i - \overline{y}_{i+1} = \overline{v}_i \tag{11}$$

for some $\overline{v}_i \in N'$, where C is generated over $W(R)$ by elements of C' and $\tilde{\varphi}^s\overline{y}_i$ ($0 \le s < \beta_i$), and D is generated over $W(R)$ by elements of D' and $\varphi^r\tilde{\varphi}^{\beta_i}\overline{y}_i$ ($1 \le r \le \alpha_i$).

Put $W_{\mathbb{Q}}(R) = \mathbb{Q} \otimes W(R)$. Write $\xi' = \sum_{l=2}^{t}(m_l, n_l)$. Then the isogeny ρ' induces an isomorphism

$$W_{\mathbb{Q}}(R) \otimes M' \xrightarrow{\sim} \bigoplus_{l=2}^{t} W_{\mathbb{Q}}(R) \otimes M((m_l, n_l)). \tag{12}$$

Let $e_l \in W_{\mathbb{Q}}(R) \otimes M'$ be the highest element of $M((m_l, n_l))$, see [4], Section 3.1. Recall that the ring of endomorphisms of H_{m_l,n_l} is described as $E_l := W(\mathbb{F}_{p^{m_l+n_l}})[\theta_l]/(\theta_l^{m_l+n_l} - p)$ for a uniformizer θ_l of $\mathrm{End}(H_{m_l,n_l})$. Let $E_l(R)$ be the $W(R)$-module $W(R) \otimes E_l$ and set $E_{l,\mathbb{Q}}(R) := \mathbb{Q} \otimes E_l(R)$. We extend the action of the Frobenius σ on $W(R)$ to that on $E_{l,\mathbb{Q}}(R)$ by the rule $\theta_l^\sigma = \theta_l$. Note the $W_{\mathbb{Q}}(R)$-homomorphism

$$E_{l,\mathbb{Q}}(R) \longrightarrow W_{\mathbb{Q}}(R) \otimes M((m_l, n_l)) \tag{13}$$

defined by sending $f(\theta_l)$ to $f(\theta_l)e_l$ is isomorphic.

We have to define $\mathcal{M} = (\mathcal{P}, \mathcal{Q}, \mathcal{F}, \dot{\mathcal{F}})$. Note that $\mathcal{P}$ should have a normal decomposition $\mathcal{L} \oplus \mathcal{T}$. We will define $\mathcal{L}$ to be the $W(R)$-submodule of $W_{\mathbb{Q}}(R) \otimes (P_1 \oplus P')$ generated by elements of L'_R and $F^r\dot{F}^{\beta_i}y_i$ ($0 \le r < \alpha_i$) and $\mathcal{T}$ will be defined to be the $W(R)$-submodule of $(P_1 \oplus P') \otimes W_{\mathbb{Q}}(R)$ generated by elements of T'_R and $\dot{F}^s y_i$ ($0 \le s < \beta_i$) for $i = 1, 2, \cdots, n$ where $y_i \in (P_1 \oplus P') \otimes W_{\mathbb{Q}}(R)$ is of the form:

$$y_i = x_i + \sum_{l=2}^{t} a_{il}e_l \tag{14}$$

for some $a_{il} \in E_{l,\mathbb{Q}}(R)$, which will be chosen later so that $\mathcal{M}$ has the required properties. Here $\mathcal{M}$ is defined by $\mathcal{P} = \mathcal{L} \oplus \mathcal{T}$ and $\mathcal{Q} = \mathcal{L} \oplus I_R\mathcal{T}$ with $\mathcal{F}$ and $\dot{\mathcal{F}}$ naturally extending F and $\dot{F}$ on $M(\xi)_R$. Since M'_R contains $M(\xi')_R$, it suffices to find $a_{il} \in E_{l,\mathbb{Q}}(R)$ modulo $I_{R,\mu}E_l(R)$ for a sufficiently large natural number μ ($\ge \max\{m_i\beta_i; i \in \mathbb{Z}/u\mathbb{Z}\}$).

Let $v_i \in P'$ be a lift of $\overline{v}_i$. We define $b_{il} \in E_{l,\mathbb{Q}}(R)$ by $\sum_{l=2}^t b_{il}e_l = v_i$. It suffices to show that there exists a solution $\{a_{il}\}$ $(i \in \mathbb{Z}/u\mathbb{Z},\ 2 \leq l \leq t)$ satisfying

$$F^{\alpha_i}\dot{F}^{\beta_i}y_i - y_{i+1} \equiv \sum_{l=2}^{t} b_{il}e_l \qquad (\mathrm{mod}\ \ I_{R,\mu}M(\xi')_R). \tag{15}$$

Comparing the coefficients of e_l of the both sides of (15), we obtain

$$a_{i,l}^{\sigma^{\alpha_i+\beta_i}}\theta_l^{n_l\alpha_i - m_l\beta_i} - a_{i+1,l} \equiv b_{il} \qquad (\mathrm{mod}\ \ I_{R,\mu}E_l(R)). \tag{16}$$

for $i \in \mathbb{Z}/u\mathbb{Z}$. Since l is the same in each equation, it suffices to solve the simultaneous equations for each l. Writing a_i, b_i, n, m and θ for a_{il}, b_{il}, n_l, m_l and θ_l respectively, we have

$$a_1^{\sigma^{\sum_{i=1}^u(\alpha_i+\beta_i)}}\theta^{\sum_{i=1}^u(n\alpha_i - m\beta_i)} - a_1 \equiv r \qquad (\mathrm{mod}\ \ I_{R,\mu}E_l(R)) \tag{17}$$

for some $r \in E_{l,\mathbb{Q}}(R)$. It suffices to show that this has a solution $a_1 \in E_{l,\mathbb{Q}}(R)$ for a finite cover $\mathrm{Spec}(R)$ of S; then we get a required solution $\{a_i\}_{i=1}^u$ from (16).

Write $z := a_1$ and $\varrho := \sigma^{\sum_{i=1}^u(\alpha_i+\beta_i)}$. Note $\varrho \neq 1$ by $\alpha_i, \beta_i > 0$. We also put $\epsilon := \sum_{i=1}^u(n\alpha_i - m\beta_i)$. Then (17) is written as $z^\varrho\theta^\epsilon - z \equiv r$ (mod $I_{R,\mu}E_l(R)$). If $\epsilon > 0$, we have a solution $z = \sum_{\ell=0}^\infty(-r)^{\varrho^\ell}\theta^{\ell\epsilon}$. Also if $\epsilon < 0$, let c be a sufficient large integer such that $\theta^{-c\epsilon} \in I_{R,\mu}E_l(R)$, and we replace R so that $R^{p^c} = R$; then we have a solution $z = \sum_{\ell=1}^{c-1} r^{\varrho^{-\ell}}\theta^{-\ell\epsilon}$. Finally we consider the case $\epsilon = 0$. Write $z = \sum_{i=0}^{m+n-1} z_i\theta^i$ and $r = \sum_{i=0}^{m+n-1} r_i\theta^i$ with $z_i, r_i \in W_{\mathbb{Q}}(R)$. It suffices to solve $z_i^\varrho - z_i \equiv r_i$ (mod $I_{R,\mu}$) for each $0 \leq i < m+n$. Let ν_i be the biggest non-negative integer ν such that $r_i \in p^\nu W(R)$. We replace $\mathrm{Spec}(R)$ by its finite (purely inseparable) cover so that we have $R^{p^{-\nu}} = R$. There exist elements t_j of R for all integers $j \geq \nu_i$ such that $z_i = \sum_{j=\nu_i}^\infty {}^{V^j}[t_j]$ is a solution. Indeed, putting $z_{ij} := \sum_{j'<j} {}^{V^{j'}}[t_{j'}]$, we can find $t_{j'}$ successively so that $z_{ij}^\varrho - z_{ij} \equiv r_i$ (mod $I_{R,j}$). Let $j \geq \nu_i$ and suppose that we have already got such $t_{j'}$ for $j' < j$. Since $\varrho \neq 1$, there exists a solution $t_j \in R$ of the Artin-Schreier equation $t_j^\varrho - t_j = ({}^{V^{-j}}(r_i - z_{ij}^\varrho + z_{ij})\ \mathrm{mod}\, I_R)$. Then clearly $z_i := \sum_{j=\nu_i}^{\mu-1} {}^{V^j}[t_j]$ is a solution of $z_i^\varrho - z_i \equiv r_i$ (mod $I_{R,\mu}$). ☐

9 Proof of Proposition 7.2

Let $w \in {}^J\mathrm{W}$. Let (m_1, n_1) be the first segment of $\xi(w)$. By the existence of $\xi(w)$, there exists a p-divisible group X over an algebraically closed field k of characteristic p such that $X[p]$ is of type w and the Newton polygon of X is $\xi(w)$. Write $M = \mathbb{D}(X)$. Choose an embedding $\imath : M \to M(\xi(w))$ and let $\jmath : M(\xi(w)) \to M_{m_1,n_1}$ be the natural projection. Put $M_1 = \jmath \circ \imath(M)$. Let $f_0 : X \to X_1$ be the homomorphism of p-divisible groups corresponding to $M \to M_1$. Let X_0' be the kernel of f_0. Note X_0' is a p-divisible group. Thus we have an exact sequence of p-divisible groups

$$0 \longrightarrow X_0' \longrightarrow X \xrightarrow{f_0} X_1 \longrightarrow 0. \tag{18}$$

Lemma 9.1. *X_1 is minimal, i.e., $X_1 \simeq H_{m_1,n_1}$.*

Proof. Recall [4], Corollary 5.4, whose dual is as follows. Let λ_v be the optimal lower bound of the first Newton slopes of p-divisible groups with p-kernel type v for each $v \in {}^J\mathrm{W}$; then we have

$$\lambda_v = \min\{m/(m+n) \mid G_{v,\Omega} \overset{\exists}{\twoheadrightarrow} H_{m,n}[p]_\Omega \text{ for some alg. closed field } \Omega\}. \tag{19}$$

Note that λ_v is equal to the first slope of $\xi(v)$.

Let w and w_1 be the final types of $X[p]$ and $X_1[p]$ respectively. Since $X[p] \twoheadrightarrow X_1[p]$, i.e., $G_{w,k} \twoheadrightarrow G_{w_1,k}$, we have $\lambda_w \le \lambda_{w_1}$ by (19). By the construction of X_1, the (first) Newton slope of X_1 is λ_w; hence we have $\lambda_w \ge \lambda_{w_1}$. Thus $\lambda_w = \lambda_{w_1}$. Then (19) implies that there exists a surjective homomorphism $H_{m_1,n_1}[p]_\Omega \twoheadrightarrow G_{w_1,\Omega}$ for some $\Omega = \overline{\Omega}$. This is an isomorphism, since $H_{m_1,n_1}[p]$ and G_{w_1} have the same rank ($= m_1 + n_1$). □

We use induction on the rank of w to prove Proposition 7.2. Assume that w is not minimal. It suffices to show the case that

(*) G_w has no direct factor which is isomorphic to $H_{m_1,n_1}[p]$.

Indeed if $G_w = G_v \oplus H_{m_1,n_1}[p]$, then v is not minimal and our problem can be reduced to the case v. Hence we assume (*) from now on. Let Z and Z_1 be F-zips of $X[p]$ and $X_1[p]$ respectively. Let $\mathcal{B}$ and $\mathcal{B}_1$ be the final types of Z and Z_1 respectively. Now (*) implies that $\Omega_\infty(\mathcal{B}, \mathcal{B}_1) = \emptyset$. Consider the space $\Sigma := \mathrm{Hom}(Z, Z_1)$, which is isomorphic to $\prod_{\omega \in \Omega_o(\mathcal{B},\mathcal{B}_1)} \mathbb{K}_\omega$; hence Σ is irreducible. Let f be the universal homomorphism $Z_\Sigma \to (Z_1)_\Sigma$.

Lemma 9.2. *Let $T \to \Sigma$ be any dominant morphism of k-schemes. Then f_T is not "constant up to $\mathrm{Aut}(Z_T)$". Here we say that f_T is constant up to $\mathrm{Aut}(Z_T)$ if there exists a section $x = \mathrm{Spec}(k) \to T$ such that $f_T = (f_x)_T \circ \kappa$ for an automorphism κ of Z_T.*

Proof. Let $x = \mathrm{Spec}(k) \to T$ be any section and κ any automorphism of Z_T. Let i be the largest integer such that $\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) = \mathrm{Hom}(Z, Z_1)$. Since $\Omega(\mathcal{B}, \mathcal{B}_1)$ consists of finite slices, we have $\dim \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1) < \dim \mathrm{Hom}(Z, Z_1)$. We write $\kappa = \kappa_o + \kappa_\infty$ with $\kappa_o \in \mathrm{End}(Z)_o(T)$ and $\kappa_\infty \in \mathrm{End}(Z)_\infty(T)$. It follows from Corollary 4.3 that $(f_x)_T \circ \kappa$ is in $\mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1)(T) + (f_x)_T \circ \kappa_\infty$. Since $\mathrm{End}(Z)_\infty$ is discrete, κ_∞ factors through $\mathrm{Spec}(k)$. Hence the dimension of the scheme-theoretic image of $(f_x)_T \circ \kappa : T \to \mathrm{Hom}(Z, Z_1)$ is less than or equal to $\dim \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1)$. On the other hand, the morphism $f_T : T \to \mathrm{Hom}(Z, Z_1)$ is dominant. Hence we have $f_T \neq (f_x)_T \circ \kappa$. □

Let η denote the generic point of Σ and let w' be the type of the kernel of f_η. Let U be the open subvariety of Σ consisting of $u \in U$ such that f_u is surjective and the kernel of f_u is of type w'. Choose a finite surjective morphism $S \to U$ which trivializes $Z' := \ker \rho_U$, i.e., $Z'_S \simeq (Z_{w'})_S$.

$$0 \longrightarrow (Z_{w'})_S \longrightarrow Z_S \xrightarrow{f_S} (Z_1)_S \longrightarrow 0. \tag{20}$$

By Corollary 2.2, there exists a display M' over k such that $M'/IM' \simeq Z_{w'}$ and the Newton polygon of M' is $\xi(w')$. Choose an isogeny $M(\xi(w')) \to M'$. Put

$$\zeta := \xi(w') + (m_1, n_1). \tag{21}$$

Applying the result of §8 to (20), for a finite surjective morphism $\mathrm{Spec}(R) \to S$, we obtain an isogeny

$$\rho : \quad M(\zeta)_R \longrightarrow \mathcal{M} \tag{22}$$

with $\phi : \mathcal{M} \to (M_1)_R$ and $\theta : \mathcal{M}/I_R\mathcal{M} \simeq Z_R$ satisfying the commutative diagrams (9) and (10).

Lemma 9.3. *We have $\zeta = \xi(w)$.*

Proof. Since $\mathcal{M}$ has Newton polygon ζ and p-kernel type w. Hence we have $\zeta \prec \xi(w)$ by the definition of $\xi(w)$. Let w'_0 be the type of $X'_0[p]$. Since $w'_0 \subset w'$, we have $\xi(w'_0) \prec \xi(w')$ by Corollary 6.2. Note the Newton polygon $\xi(w)$ of X is equal to $\xi(w'_0) + (m_1, n_1)$. This is less than or equal to $\xi(w') + (m_1, n_1) = \zeta$. □

It remains to show

Lemma 9.4. *ρ is not constant.*

Proof. Applying Lemma 9.2 to $T := \mathrm{Spec}(R) \to \Sigma$, we have that f_R is not constant up to $\mathrm{Aut}(Z_R)$. It follows from the diagram (10) that $\overline{\phi}$ is non-constant; hence so is ϕ. Then we have the lemma by the diagram (9). □

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