The supremum of Newton polygons of p-divisible groups with a given p-kernel type

Shushi Harashita

April 6, 2011

Dedicated to Professor Takayuki Oda on his 60th birthday

Abstract

In this paper we show that there exists the supremum of Newton polygons of p-divisible groups with a given p-kernel type, and provide an algorithm determining it.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. We are concerned with estimating the isogeny type (=Newton polygon, cf. [11]) of a p-divisible group X over k from its p-kernel $X[p]$. In this paper we give an optimal estimation.

We fix once for all, non-negative integers c and d with $r := c + d > 0$. Let $\mathrm{W}(= \mathrm{W}_r)$ be the Weyl group of the general linear group GL_r. In the usual sense, we identify W and $\mathrm{Aut}(\{1, \ldots, r\})$. Let $s_i \in \mathrm{W}$ be the simple reflection $(i, i + 1)$ for $i = 1, \ldots, r - 1$. Let $\mathrm{S} = \{s_1, \ldots, s_{r-1}\}$ and set $\mathrm{J} := \mathrm{S} \setminus \{s_d\}$. Let W_J be the subgroup of W generated by elements of J. We denote by ${}^\mathrm{J}\mathrm{W}$ the set of $(\mathrm{J}, \emptyset)$-reduced elements of W (cf. [1], Chap. IV, Ex. §1, 3), which are the shortest representatives of $\mathrm{W}_\mathrm{J} \setminus \mathrm{W}$. A classification theory of BT_1's by Kraft, Oort, Moonen and Wedhorn says that the set of the isomorphism classes of BT_1's with tangent-dimension d and length r is bijective to the set ${}^\mathrm{J}\mathrm{W}$. Note that ${}^\mathrm{J}\mathrm{W}$ has a natural ordering $\subset$ introduced and investigated by He [7], also see Wedhorn [22] (we shall give a short review: Theorem 3.6).

Let us explain our main results: Theorem 1.1 combined with Corollary 2.2. Let w be any element of ${}^\mathrm{J}\mathrm{W}$. In Corollary 2.2 we show that there exists the supremum $\xi(w)$ of Newton polygons of p-divisible groups with p-kernel type w:

- every p-divisible group whose p-kernel is of type w has Newton polygon $\prec \xi(w)$;
- there exists a p-divisible group X such that $X[p]$ is of type w and the Newton polygon of X equals $\xi(w)$.

Theorem 1.1 below gives us a combinatorial algorithm determining $\xi(w)$, see Remark 5.1. For a Newton polygon ζ, let $\mu(\zeta) \in {}^\mathrm{J}\mathrm{W}$ denote the type of the p-kernel of the minimal p-divisible group $H(\zeta)$ having Newton polygon ζ (cf. [15] and also a review [4], §3).

Theorem 1.1. *$\xi(w)$ is the biggest one of the Newton polygons ζ with $\mu(\zeta) \subset w$.*

We shall see in §5 that this theorem follows from Proposition 5.2. The last two sections are devoted to the proof of Proposition 5.2. The theorem is an unpolarized analogue of [6], Corollary II. For a more effective algorithm determining the first/last slope of $\xi(w)$, see [3], Theorem 4.1 for the polarized case and [4], Corollary 1.3 for the unpolarized case. In the polarized case, the existence of the supremum $\xi(w)$ follows from the fact that any Ekedahl-Oort stratum on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties is irreducible if it is not contained in the supersingular locus ([2], Theorem 11.5). An obstruction in the unpolarized case has been the absence of a good moduli space like $\mathcal{A}_g$. However using Vasiu's $\mathbb{T}_m$-action instead, we have (Lemma 2.1) that there exists an irreducible catalogue of p-divisible groups with a fixed p-kernel type; this clearly shows the existence of $\xi(w)$. Then Theorem 1.1 can be shown by a similar argument as in [6] (which is relatively easier than the polarized case). Finally we mention a different approach announced by Viehmann [21], who seems to have generalized our results in terms of the loop groups of split reductive groups, making use of results on affine Deligne-Lusztig varieties.

Terminology

We naturally identify the category of affine schemes with the opposite category to the category of commutative rings. We fix once for all a rational prime p. In this paper we freely use a part of Zink's result [23], Theorem 9, which says that for a commutative ring R of finite type over a field of characteristic p, there exists a categorical equivalence from the category of formal p-divisible groups over R to that of nilpotent displays over R, where we follow the terminology of [10] for displays and nilpotent displays.

2 A catalogue of p-divisible groups with a given type

Let k be an algebraically closed field of characteristic p. Let $(P, Q, F, \dot{F})$ be a display over k, and $P = L \oplus T$ be a normal decomposition ([23], Introduction), where L and T are free $W(k)$-modules. Let c and d be the ranks of L and T respectively. Let $\mathrm{G} = \mathrm{GL}(P)$ be the general linear group over $W(k)$ of degree $r = c + d$. Let H be the parahoric subgroup of G stabilizing Q, which is a connected smooth affine group scheme over $W(k)$. Let $\mathcal{D}_m$ and $\mathcal{H}_m$ be connected smooth affine group schemes over k such that $\mathcal{D}_m(R) = \mathrm{G}(W_m(R))$ and $\mathcal{H}_m(R) = \mathrm{H}(W_m(R))$ respectively, see [19], 2.1.4 for more details. For any truncated Barsotti-Tate group of level m with codimension c and dimension d, there exists a $g \in \mathcal{D}_m$ such that its Dieudonné module is isomorphic to the $W_m(k)$-module $P/p^m P$ with the Frobenius and the Verschiebung defined by gF and Vg^{-1} respectively. Vasiu introduced an action:

$$\mathbb{T}_m : \quad \mathcal{H}_m \times_k \mathcal{D}_m \longrightarrow \mathcal{D}_m, \tag{1}$$

and showed in [19], 2.2.2 that the set of $\mathbb{T}_m$-orbits is naturally bijective to the set of isomorphism classes of truncated Barsotti-Tate groups of level m over k with codimension c and dimension d. Let $\mathbf{BT}_m(k)$ be the set of isomorphism classes of truncated Barsotti-Tate groups of level m over k with codimension c and dimension d. We have

Lemma 2.1. *For any $u \in \mathbf{BT}_m(k)$, there exists an irreducible catalogue of p-divisible groups with p^m-kernel type u, i.e., there exists a family $\mathcal{X} \to S$ of p-divisible groups such that*

(1) *for any geometric point $s \in S$, the p^m-kernel of the fiber $\mathcal{X}_s$ is of type u;*

(2) *For any p-divisible group X over k with p^m-kernel type u, there exists an $s \in S(k)$ such that $X \simeq \mathcal{X}_s$;*

(3) *S is irreducible and of finite type over k.*

Proof. It suffices to consider the case that u has no étale part, since every (truncated) Barsotti-Tate group over k is the direct sum of its local part and its étale part and the decomposition is compatible with truncations. Let N be an integer $\geq m$ so that $X[p^N] \simeq$

$Y[p^N]$ implies $X \simeq Y$ for any p-divisible groups X and Y over k (cf. [16], 1.7 and [20]). Let π be the natural map $\mathcal{D}_N \to \mathcal{D}_m$. Let $\mathcal{D}$ be the (group) scheme over k such that $\mathcal{D}(R) = \mathrm{GL}(W(R)$ for any k-algebra R, and let τ be a section of $\mathcal{D} \to \mathcal{D}_N$ as a morphism of schemes. Let $\mathbb{O}_u$ be the $\mathbb{T}_m$-orbit associated to u. Since $\mathcal{H}_m$ is irreducible, $\mathbb{O}_u$ is irreducible. Since π is smooth with connected fibers, $\pi^{-1}(\mathbb{O}_u)$ is also irreducible. Let S be the image of $\pi^{-1}(\mathbb{O}_u)$ by τ. Then S is irreducible and of finite type over k. By [23], Theorem 9, we have a p-divisible group $\mathcal{X}$ over S. Clearly $\mathcal{X}$ satisfies the required properties. □

Corollary 2.2. *There exists the supremum of Newton polygons of p-divisible groups with the given p^m-kernel type.*

Proof. Let $\mathcal{X} \to S$ be the family as in the lemma above. Let η be the generic point of S. It follows from Grothendieck and Katz ([8], Th. 2.3.1 on p. 143) that the Newton polygon of $\mathcal{X}_\eta$ is the supremum of Newton polygons of p-divisible groups with a given p^m-kernel type. □

3 Preliminaries on F-zips

In this section, we collect some basic facts on F-zips which we shall use later on.

We first recall the definition ([13], (1.5)) of F-zip in a particular case. Let S be a scheme of characteristic p. Let σ denote the absolute Frobenius on S. For any $\mathcal{O}_S$-module M we write $M^{(p)} = \mathcal{O}_S \otimes_{\sigma,\mathcal{O}_S} M$.

Definition 3.1. *An F-zip over S is a quintuple $Z = (N, C, D, \varphi, \dot\varphi)$ consisting of locally free $\mathcal{O}_S$-module N and $\mathcal{O}_S$-submodules C, D of N which are locally direct summands of N, and σ-linear homomorphisms $\varphi : N/C \to D$ and $\dot\varphi : C \to N/D$ whose $\mathcal{O}_S$-linearizations $\varphi^\sharp : (N/C)^{(p)} \to D$ and $\dot\varphi^\sharp : C^{(p)} \to N/D$ are isomorphisms. If S is connected, we define the height of Z to be the rank of N and the type of Z to be a map from $\{0, 1\}$ to $\mathbb{Z}_{\geq 0}$ sending 0 to $\operatorname{rk} D$ and 1 to $\operatorname{rk} C$; we will simply write the type as $(\operatorname{rk} D, \operatorname{rk} C)$.*

If S is the spectrum of a perfect field K, then the (covariant) Dieudonné functor $\mathbb{D}$ makes an equivalence from the category of BT_1's over K to that of F-zips over K. The F-zip $(N, C, D, \varphi, \dot\varphi)$ associated to a BT_1-group G is given by $N = \mathbb{D}(G)$ with $C = VN$ and $D = FN$, and φ and $\dot\varphi^{-1}$ are naturally induced by F and V respectively.

Let k be an algebraically closed field of characteristic p. Let c, d, r and ${}^{\mathrm{J}}\mathrm{W}$ be as in Introduction. As shown by Kraft, Oort, Moonen and Wedhorn, there exists a bijection from $\mathbf{BT}_1(k)$ to ${}^{\mathrm{J}}\mathrm{W}$ (this statement is due to [13]; also see [9], [12] and [17]). This classification is based on the fact that for any BT_1-group G over k, there uniquely exists $w \in {}^{\mathrm{J}}\mathrm{W}$ such that G is isomorphic to G_w defined below. To a $w \in {}^{\mathrm{J}}\mathrm{W}$ we associate a pair (B, δ), called a *final type*, cf. [4], Definition 2.6, where B is a totally ordered set $\{b_1 < \ldots < b_r\}$ and δ is a map $B \to \{0, 1\}$ defined by $\delta(b_i) = 1 \Leftrightarrow w(i) \leq d$. There uniquely exists an automorphism $\pi = \pi_\delta$ of B such that $\pi(b') > \pi(b) \Leftrightarrow \delta(b') > \delta(b)$ for any $b' < b$. We define G_w so that its F-zip $Z_w = (N, C, D, \varphi, \dot\varphi)$ is given by $N = \bigoplus_{b \in B} kb$ (i.e., the k-vector space with basis indexed by B) and $C = \bigoplus_{\delta(b)=1} kb$, and $D = \bigoplus_{\delta(b)=0} k\pi(b)$ and φ, $\dot\varphi$ are defined by $\varphi(b) = \pi(b)$ for b with $\delta(b) = 0$ and $\dot\varphi(b) = \pi(b)$ for b with $\delta(b) = 1$.

Next we review a description of homomorphisms of F-zips over k for the reader's convenience (cf. [14], §2 and [12], §4 and also see [4], § 4.3), and show some facts used later on. Let Z_1 and Z_2 be two F-zips over k. Let w_1 and w_2 be the types of Z_1 and Z_2 respectively and let $\mathcal{B}_1 = (B_1, \delta_1)$ and $\mathcal{B}_2 = (B_2, \delta_2)$ be their final types. Set $\pi_1 = \pi_{\delta_1}$ and $\pi_2 = \pi_{\delta_2}$. A *finite slice* ω is a subset of $B_1 \times B_2$ of the form $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ with $|\omega| = \ell$ for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying **(a)** $\delta_1(s_1) = 1$ and $\delta_2(s_2) = 0$, **(b)** $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \leq i < \ell$ and **(c)** $\delta_1(\pi_1^\ell(s_1)) = 0$ and $\delta_2(\pi_2^\ell(s_2)) = 1$. We denote by $\Omega_o = \Omega_o(\mathcal{B}_1, \mathcal{B}_2)$ the set of finite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$. An *infinite slice* ω is a subset of $B_1 \times B_2$ of the form $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ with $|\omega| = \ell$ for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying **(a)**

$s_1 = \pi_1^\ell(s_1)$ and $s_2 = \pi_2^\ell(s_2)$, **(b)** $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \le i < \ell$. We denote by $\Omega_\infty = \Omega_\infty(\mathcal{B}_1, \mathcal{B}_2)$ the set of infinite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$. Set $\Omega = \Omega(\mathcal{B}_1, \mathcal{B}_2) := \Omega_o \sqcup \Omega_\infty$. For each slice ω, we define a group scheme $\mathbb{K}_\omega$ over k to be the additive group $\mathbb{G}_a$ over k if $\omega \in \Omega_o$ and to be $\mathrm{Ker}(F^{|\omega|} - \mathrm{id} : \ \mathbb{G}_a \to \mathbb{G}_a)$ if $\omega \in \Omega_\infty$. Let S be a k-scheme. Let $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \le i \le \ell\}$ be a slice with $|\omega| = \ell$. For an element $r \in \omega$, we denote by $\varepsilon(r)$ $(= \varepsilon_\omega(r))$ the integer ε with $0 \le \varepsilon < \ell$ satisfying $r = (\pi_1^{\varepsilon+1}(s_1), \pi_2^{\varepsilon+1}(s_2))$. For $a \in \mathbb{K}_\omega(S)$, we define a map

$$\mathrm{st}_{\omega,a} : \quad B_1 \times B_2 \longrightarrow \mathbb{K}_\omega(S) \tag{2}$$

by sending $r \in \omega$ to $a^{p^{\varepsilon(r)}}$ and $r \notin \omega$ to 0. The functor, from the category of k-schemes to the category of commutative groups, sending S to $\mathit{Hom}_S(Z_{1,S}, Z_{2,S})$ is represented by a group scheme $\mathrm{Hom}(Z_1, Z_2)$ over k; moreover there is an isomorphism as group schemes over k:

$$\Lambda : \quad \bigoplus_{\omega \in \Omega} \mathbb{K}_\omega \xrightarrow{\ \sim\ } \mathrm{Hom}(Z_1, Z_2). \tag{3}$$

Indeed, for $* = 1, 2$, we write $B_* = \{b_1^{(*)} < \cdots < b_{r_*}^{(*)}\}$ and also write $Z_* = (N_*, C_*, D_*, \varphi_*, \dot{\varphi}_*)$ with $N_* = \bigoplus_{i=1}^{r_*} k b_i^{(*)}$. Let S be any k-scheme. An $\mathcal{O}_S$-homomorphism $\mu : N_{1,S} \to N_{2,S}$, say $\mu(b_i^{(1)}) = \sum_j r_{ij} b_j^{(2)}$ with $r_{ij} \in \Gamma(S, \mathcal{O}_S)$ gives an element of $\mathit{Hom}_S(Z_{1,S}, Z_{2,S})$ if and only if r_{ij} is of the form $\sum_{\omega\in\Omega} \mathrm{st}_{\omega,a}(b_{ij})$ for a certain $a \in \mathbb{K}_\omega(S)$, where $b_{ij} = (b_i^{(1)}, b_j^{(2)}) \in B_1 \times B_2$. From now on we identify $\mathrm{Hom}(Z_1, Z_2)$ with $\bigoplus_{\omega\in\Omega} \mathbb{K}_\omega$. The connected component of zero in a commutative group scheme H will be denoted by H_o. Then $\mathrm{Hom}(Z_1, Z_2)_o$ is the product of $\mathbb{K}_\omega$ for $\omega \in \Omega_o$. We write $\mathrm{Hom}(Z_1, Z_2)_\infty$ for $\bigoplus_{\omega\in\Omega_\infty} \mathbb{K}_\omega$.

It is straightforward to prove

Lemma 3.2. *Let Z_1, Z_2, Z_3 be F-zips over k. The composition map*

$$\mathrm{Hom}(Z_1, Z_2) \times \mathrm{Hom}(Z_2, Z_3) \longrightarrow \mathrm{Hom}(Z_1, Z_3)$$

sends the pair of (ω_1, a_1) and (ω_2, a_2) (i.e., $\omega_i \in \Omega(\mathcal{B}_i, \mathcal{B}_{i+1})$ and $a_i \in \mathbb{K}_{\omega_i}$ for $i = 1, 2$) to $\sum_{\tilde\omega} (\omega, a_1^{p^e} a_2^{p^f})$, where the sum is over $\pi_1 \times \pi_2 \times \pi_3$-orbits $\tilde\omega$ in $\omega_1 \times_{B_2} \omega_2$ and $\omega = \mathrm{pr}_{13}(\tilde\omega)$ and e is the minimal element of $\varepsilon_{\omega_1}(\mathrm{pr}_{12}(\tilde\omega))$ and f is the minimal element of $\varepsilon_{\omega_2}(\mathrm{pr}_{23}(\tilde\omega))$. Here we denote by pr_{ij} the projections $B_1 \times B_2 \times B_3 \to B_i \times B_j$ for $1 \le i < j \le 3$.

The next lemma shows that the ring scheme $\mathrm{End}(Z)_o$ consists of nilpotent endomorphisms.

Lemma 3.3. *Let $\omega \in \Omega_o(\mathcal{B}, \mathcal{B})$. Let (b, b') be an element of ω. Then we have $b > b'$.*

Proof. By the definition of finite slice, $\nu(b) := \sum_{l\in\mathbb{N}} \delta(\pi^{-l}(b)) 2^{-l}$ is greater than $\nu(b') := \sum_{l\in\mathbb{N}} \delta(\pi^{-l}(b')) 2^{-l}$. Then [5], Proposition 4.7 shows $b > b'$. □

For later use, we look at the action of $\mathrm{End}(Z)_o$ on $\mathrm{Hom}(Z, Z_1)$. Let $\Omega^i(\mathcal{B}, \mathcal{B}_1)$ be the subset of $\Omega(\mathcal{B}, \mathcal{B}_1)$ consisting of $\omega \in \Omega(\mathcal{B}, \mathcal{B}_1)$ with $\mathrm{pr}(\omega) \subset \{b_i, \ldots, b_r\}$, where pr is the projection $B \times B_1 \to B$. We define a subgroup scheme $\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1)$ of $\mathrm{Hom}(Z, Z_1)$ by

$$\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) = \bigoplus_{\omega \in \Omega^i(\mathcal{B}, \mathcal{B}_1)} \mathbb{K}_\omega. \tag{4}$$

From the lemmas above, we have

Corollary 3.4. *The composition map induces*

$$\mathrm{End}(Z)_o \times \mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) \longrightarrow \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1).$$

At the end of this section, we recall Wedhorn's result ([22]) on specializations of F-zips.

Definition 3.5. *Let w and w' be elements of ${}^{\mathrm{J}}\mathrm{W}$. We say $w \subset w'$ if there exists an F-zip over an irreducible scheme of characteristic p such that the generic fiber is of type w' and there exists a closed point such that the fiber over that point is of type w.*

Let $x = w_0^{\mathrm{J}} : \mathrm{W} \to \mathrm{W}$ be the map sending i to $i+c$ if $i \leq d$ and i to $i-d$ if $i > d$. Define $\delta : \mathrm{W} \to \mathrm{W}$ by $\delta(u) = x \cdot u \cdot x^{-1}$.

Theorem 3.6 ([22])**.** *Let $w, w' \in {}^{\mathrm{J}}\mathrm{W}$. We have $w \subset w'$ if and only if there exists $u \in \mathrm{W}_{\mathrm{J}}$ such that $u^{-1}w\delta(u)$ is less than or equal to w' with respect to the Bruhat order.*

4 Lifting of F-zips

Let R be a commutative ring of characteristic p. Let ^{F} and ^{V} denote the Frobenius and Verschiebung on $W(R)$. Write $I_R := {}^V W(R)$. Let $M = (P, Q, F, \dot{F})$ be a display over R. One can associate to M an F-zip $M/I_R M$, which is defined as follows. Let $P = L \oplus T$ be a normal decomposition of P with $Q = L \oplus I_R T$ (cf. [23], Introduction). We define $M/I_R M$ to be $(N, C, D, \varphi, \dot{\varphi})$ where $N = P/I_R P$ and $C = Q/I_R P \simeq L/I_R L$, and D is the submodule of N generated by the image of $F : T \to P \to N$, and φ and $\dot{\varphi}$ are canonically induced by F and $\dot{F}$ respectively.

Lemma 4.1. *Let Z be an F-zip over S. Let s be any closed point of S. Let M be a display over s. There is an open affine subscheme $U = \mathrm{Spec}(R)$ of S with $s \in U$ and a display $\mathcal{M}$ over R such that $\mathcal{M}/I_R\mathcal{M} \simeq Z_R$ and $\mathcal{M}_s \simeq M$.*

Proof. Write $Z = (N, C, D, \varphi, \dot{\varphi})$. Let U be an affine open subscheme of S containing s over which C and D are direct summands of N, say $N = C \oplus E$, and C, D and E are free. Write $U = \mathrm{Spec}(R)$ and $s = \mathrm{Spec}(R/\mathfrak{m})$. We replace S by U. We have an ^{F}-linear homomorphism

$$\phi : \quad C \oplus E \xrightarrow{\ \sim\ } C \oplus N/C \xrightarrow{\dot{\varphi}\oplus\varphi} N/D \oplus D \xrightarrow{\ \sim\ } N.$$

Let $\mathcal{L}$ and $\mathcal{T}$ be free $W(R)$-modules such that $\mathcal{L}/I_R\mathcal{L} \simeq C$ and $\mathcal{T}/I_R\mathcal{T} \simeq E$. Put $\mathcal{P} = \mathcal{L} \oplus \mathcal{T}$ and $\mathcal{Q} = \mathcal{L} \oplus I_R\mathcal{T}$. Let $M = (P, Q, F, \dot{F})$ and $L \oplus T$ a normal decomposition of P, and let Φ_0 be $\dot{F} \oplus F : L \oplus T \to P$ obtained in [23], Lemma 9; one can identify L and T with $\mathcal{L}/W(\mathfrak{m})\mathcal{L}$ and $\mathcal{T}/W(\mathfrak{m})\mathcal{T}$ respectively.

Since the canonical map from $\mathrm{GL}_r(W(R))$ to the fiber product of $\mathrm{GL}_r(R) \to \mathrm{GL}_r(R/\mathfrak{m})$ and $\mathrm{GL}_r(W(R/\mathfrak{m})) \to \mathrm{GL}_r(R/\mathfrak{m})$ is clearly surjective, there exists an ^{F}-linear homomorphism

$$\Phi : \quad \mathcal{L} \oplus \mathcal{T} \longrightarrow \mathcal{P}$$

such that $(\Phi \mod I_R) = \phi$ and $(\Phi \mod W(\mathfrak{m})) = \Phi_0$. Set $\mathcal{F} = \Phi \circ ({}^V 1 \oplus \mathrm{id}) : \mathcal{L} \oplus \mathcal{T} \to \mathcal{P}$ and define $\dot{\mathcal{F}} : \mathcal{L} \oplus I_R\mathcal{T} \to \mathcal{P}$ by sending $l + {}^V wt$ to $\Phi(l) + w\Phi(t)$ for every $l \in \mathcal{L}$, $t \in \mathcal{T}$ and $w \in W(R)$. Then we have a display $\mathcal{M} = (\mathcal{P}, \mathcal{Q}, \mathcal{F}, \dot{\mathcal{F}})$, which satisfies the required properties. □

For $v \in {}^{\mathrm{J}}\mathrm{W}$, let $\xi(v)$ be the Newton polygon introduced in §1.

Corollary 4.2. *Let w and w' be elements of ${}^{\mathrm{J}}\mathrm{W}$. If $w \subset w'$, then we have $\xi(w) \prec \xi(w')$.*

Proof. Assume $w \subset w'$, i.e., there exists an F-zip Z over an irreducible scheme S such that the type of the fiber of the generic point η is w' and the type of the fiber of a special point s is w. By the definition of $\xi(w)$, there exists a display M over s such that the Newton polygon of M is $\xi(w)$. Applying Lemma 4.1 to Z and M, there exist an open affine subscheme $U = \mathrm{Spec}(R)$ of S containing s and a display $\mathcal{M}$ over R such that $\mathcal{M}/I_R\mathcal{M} \simeq Z_R$ and $\mathcal{M}_s \simeq M$. It follows from Grothendieck-Katz ([8], Th. 2.3.1 on p. 143) that $\xi(w)$ is less than or equal to the Newton polygon, say ζ, of $\mathcal{M}_\eta$. By the definition of $\xi(w')$, we have $\zeta \prec \xi(w')$. □

5 A reduction of the problem

Let w be any element of ${}^{\mathrm{J}}\mathrm{W}$. The purpose of this paper is to prove Theorem 1.1:

$$\xi(w) = \max_{\prec}\{\zeta \mid \mu(\zeta) \subset w\}, \tag{5}$$

where ζ is over Newton polygons $\sum(m_i, n_i)$ with $\sum m_i = d$ and $\sum n_i = c$ (see §1 for the definitions of $\xi(w)$ and $\mu(\zeta)$).

Remark 5.1. *This gives, thanks to Theorem 3.6, a purely combinatorial algorithm determining $\xi(w)$ for a given w. See [5], Corollary 4.8 for a way to compute $\mu(\zeta)$.*

We first prove that Theorem 1.1 follows from the next proposition. The subsequent sections are devoted to the proof of this proposition.

Proposition 5.2. *Assume that w is not minimal. Then there exist a scheme S of finite type over k with $\dim S \geq 1$, a p-divisible group $\mathcal{X}$ over S and a non-constant family of isogenies*

$$H(\xi(w))_S \longrightarrow \mathcal{X} \tag{6}$$

over S such that the isomorphism type of $\mathcal{X}_s[p]$ is w for every geometric point s of S.

Proof of (Proposition 5.2 ⇒ Theorem 1.1). We first claim that Theorem 1.1 is equivalent to

$$\mu(\xi(w)) \subset w. \tag{7}$$

Clearly Theorem 1.1 implies (7). Suppose (7). Put $\Xi = \{\zeta \mid \mu(\zeta) \subset w\}$. We want to show that $\xi(w)$ is the biggest element of Ξ. Clearly (7) says $\xi(w) \in \Xi$. Let ζ be any element of Ξ. Then we have $\xi(\mu(\zeta)) \prec \xi(w)$ by Corollary 4.2. Note that we have $\xi(\mu(\zeta)) = \zeta$ by [15], (1.2) Theorem. Thus we have $\zeta \prec \xi(w)$.

Let us prove (7) under the assumption that Proposition 5.2 holds. We first consider the case that w is minimal, say $w = \mu(\zeta)$ the type of $H(\zeta)[p]$. Then we have $\xi(w) = \zeta$ by [15] (1.2), and therefore we have $\mu(\xi(w)) = w$; hence (7) holds in this case. Assume that w is not minimal. Let $\mathcal{M}$ be the moduli space of quasi-isogenies $H(\xi(w)) \to Y$ of p-divisible groups, see [18], Chapter 2. Let $\mathcal{I}$ be an irreducible component of $\mathcal{M}_{\mathrm{red}}$ containing the generic point of the family (6). Note that $\mathcal{I}$ is projective ([18], Proposition 2.32). Let $\mathcal{S}_w(\mathcal{I})$ be the locally closed subvariety consisting of isogenies $H(\xi(w)) \to X$ where $X[p]$ is of type w. It is known that $\mathcal{S}_w(\mathcal{I})$ is quasi-affine (cf. [19], 1.2 (g)). By the assumption, we have $\dim \mathcal{S}_w(\mathcal{I}) \geq 1$. Hence there exists $w' \in {}^{\mathrm{J}}\mathrm{W}$ such that $w' \subsetneq w$ and $\xi(w') = \xi(w)$. This shows in particular that any final type v with $\xi(v) = \xi(w)$ which is "minimal w.r.t. $\subset$" is minimal. We use induction on w with respect to $\subset$. We assume $\mu(\xi(w')) \subset w'$. Then we have $\mu(\xi(w)) = \mu(\xi(w')) \subset w' \subset w$. □

6 Extensions by a minimal p-divisible group

Let ξ be a Newton polygon. Let $\varrho = (m_1, n_1)$ be a segment of ξ, i.e., $m_1/(m_1+n_1)$ is a slope of ξ and $\gcd(m_1, n_1) = 1$. Let ξ' be the Newton polygon such that $\xi = \xi' + \varrho$. Let $Z_1 = Z(\varrho)_S$, where $Z(\varrho)$ is the F-zip of $H(\varrho)[p]$ over k. Let

$$0 \longrightarrow Z' \longrightarrow Z \xrightarrow{\ f\ } Z_1 \longrightarrow 0 \tag{8}$$

be a short exact sequence of F-zips over a reduced k-scheme S. Write $Z = (N, C, D, \varphi, \dot\varphi)$ and $Z_1 = (N_1, C_1, D_1, \varphi_1, \dot\varphi_1)$ and so on. That f is surjective means that $f : N \to N_1$ and $f : C \to C_1$ are surjective, and also the injectivity is the dual notion of this surjectivity. Let $M' = (P', Q', F', \dot F')$ be any display lifting Z' with an isogeny $\rho' : M(\xi')_S \to M'$, where $M(\xi')$ is the display of $H(\xi')$. Let $P' = L' \oplus T'$ be a normal decomposition.

Proposition 6.1. *For any closed point $s \in S$, there exist an open affine subscheme U of S with $s \in U$ and a finite surjective morphism* $\mathrm{Spec}(R) \to U$ *such that there exist a display $\mathcal{M}$ over R with an isogeny $\rho : M(\xi)_R \to \mathcal{M}$ and a surjective homomorphism $\phi : \mathcal{M} \to (M_1)_R := M(\varrho)_R$ with kernel M'_R and an isomorphism $\theta : \mathcal{M}/I_R\mathcal{M} \to Z_R$ such that we have the commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\xi')_R & \longrightarrow & M(\xi)_R & \xrightarrow{\mathrm{pr}} & M(\varrho)_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \| & & \\ 0 & \longrightarrow & M'_R & \longrightarrow & \mathcal{M} & \xrightarrow{\phi} & (M_1)_R & \longrightarrow & 0 \end{array} \tag{9}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_R/I_R M'_R & \longrightarrow & \mathcal{M}/I_R\mathcal{M} & \xrightarrow{\overline{\phi}} & (M_1)_R/I_R(M_1)_R & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \theta}\downarrow{\scriptstyle \simeq} & & \| & & \\ 0 & \longrightarrow & Z'_R & \longrightarrow & Z_R & \xrightarrow{f_R} & (Z_1)_R & \longrightarrow & 0. \end{array} \tag{10}$$

Proof. Let $u = \min\{m_1, n_1\}$. Recall [4], Lemma 3.3 that the Dieudonné module $M(\varrho)$ is generated over the Dieudonné ring by X_i ($i \in \mathbb{Z}/u\mathbb{Z}$) and all relations are generated by $F^{\alpha_i}X_i - V^{\beta_{i+1}}X_{i+1} = 0$ for some non-negative integers α_i, β_i. Put $x_i := V^{\beta_i}X_i$. Set $\beta = \max\{\beta_i\}$.

Let s be any closed point of S. Let $U = \mathrm{Spec}(R)$ be an affine open subscheme of S containing s. We may replace S by U. We choose a lift $\overline{Y}_i \in N$ of $\overline{X}_i \in N_1$ for each $i \in \mathbb{Z}/u\mathbb{Z}$. Let ψ be the composition of $N \to N/D$ and $(\dot{\varphi}^\sharp)^{-1} : N/D \to C^{(p)}$. After replacing R with R' such that $(R')^{p^\beta} = R$, we can find $\overline{y}_{i,j} \in C$ lifting $V^j\overline{X}_i$ for $0 \leq j \leq \beta_i$ such that the composition $\psi^{(p^{j-1})} \circ \cdots \circ \psi^{(p)} \circ \psi$ sends $\overline{Y}_i$ to $1 \otimes \overline{y}_{i,j} \in R \otimes_{F^j,R} C$. We put $\overline{y}_i := \overline{y}_{i,\beta_i}$. After replacing R by an open affine subscheme over which N is free, we can find a section of $N \to N/D$, defining a lift $\tilde{\varphi} : C \to N$ of $\dot{\varphi} : C \to N/D$, such that $\tilde{\varphi}^{\beta_i - j}(\overline{y}_i) = \overline{y}_{i,j}$. It follows from the exact sequence (8) that N is generated by elements of N' and $\tilde{\varphi}^s\overline{y}_i$ ($0 \leq s < \beta_i$) and $\varphi^r\tilde{\varphi}^{\beta_i}\overline{y}_i$ ($0 \leq r < \alpha_i$) with relations

$$\varphi^{\alpha_i}\tilde{\varphi}^{\beta_i}\overline{y}_i - \overline{y}_{i+1} = \overline{v}_i \tag{11}$$

for some $\overline{v}_i \in N'$, where C is generated over $W(R)$ by elements of C' and $\tilde{\varphi}^s\overline{y}_i$ ($0 \leq s < \beta_i$), and D is generated over $W(R)$ by elements of D' and $\varphi^r\tilde{\varphi}^{\beta_i}\overline{y}_i$ ($1 \leq r \leq \alpha_i$).

Put $W_{\mathbb{Q}}(R) = \mathbb{Q} \otimes W(R)$. Write $\xi' = \sum_{l=2}^t (m_l, n_l)$. Then the isogeny ρ' induces an isomorphism

$$W_{\mathbb{Q}}(R) \otimes M' \xrightarrow{\sim} \bigoplus_{l=2}^t W_{\mathbb{Q}}(R) \otimes M((m_l, n_l)). \tag{12}$$

Let $e_l \in W_{\mathbb{Q}}(R) \otimes M'$ be the highest element of $M((m_l, n_l))$, see [4], Section 3.1. Recall that the ring of endomorphisms of H_{m_l,n_l} is described as $E_l := W(\mathbb{F}_{p^{m_l+n_l}})[\theta_l]/(\theta_l^{m_l+n_l} - p)$ for a uniformizer θ_l of $\mathrm{End}(H_{m_l,n_l})$. Let $E_l(R)$ be the $W(R)$-module $W(R) \otimes E_l$ and set $E_{l,\mathbb{Q}}(R) := \mathbb{Q} \otimes E_l(R)$. We extend the action of the Frobenius σ on $W(R)$ to that on $E_{l,\mathbb{Q}}(R)$ by the rule $\theta_l^\sigma = \theta_l$. Note the $W_{\mathbb{Q}}(R)$-homomorphism

$$E_{l,\mathbb{Q}}(R) \longrightarrow W_{\mathbb{Q}}(R) \otimes M((m_l, n_l)) \tag{13}$$

defined by sending $f(\theta_l)$ to $f(\theta_l)e_l$ is an isomorphism.

We have to define $\mathcal{M} = (\mathcal{P}, \mathcal{Q}, \mathcal{F}, \dot{\mathcal{F}})$. Note that $\mathcal{P}$ should have a normal decomposition $\mathcal{L} \oplus \mathcal{T}$. We will define $\mathcal{L}$ to be the $W(R)$-submodule of $W_{\mathbb{Q}}(R) \otimes (P_1 \oplus P')$ generated by elements of L'_R and $F^r\dot{F}^{\beta_i}y_i$ $(0 \le r < \alpha_i)$ and $\mathcal{T}$ will be defined to be the $W(R)$-submodule of $(P_1 \oplus P') \otimes W_{\mathbb{Q}}(R)$ generated by elements of T'_R and $\dot{F}^s y_i$ $(0 \le s < \beta_i)$ for $i = 1, 2, \cdots, n$ where $y_i \in (P_1 \oplus P') \otimes W_{\mathbb{Q}}(R)$ is of the form:

$$y_i = x_i + \sum_{l=2}^{t} a_{il} e_l \tag{14}$$

for some $a_{il} \in E_{l,\mathbb{Q}}(R)$, which will be chosen later so that $\mathcal{M}$ has the required properties. Here $\mathcal{M}$ is defined by $\mathcal{P} = \mathcal{L} \oplus \mathcal{T}$ and $\mathcal{Q} = \mathcal{L} \oplus I_R\mathcal{T}$ with $\mathcal{F}$ and $\dot{\mathcal{F}}$ naturally extending F and $\dot{F}$ on $M(\xi)_R$. Since M'_R contains $M(\xi')_R$, it suffices to find $a_{il} \in E_{l,\mathbb{Q}}(R)$ modulo $I_{R,\mu}E_l(R)$ for a sufficiently large natural number μ $(\ge \max\{m_i\beta_i; i \in \mathbb{Z}/u\mathbb{Z}\})$.

Let $v_i \in P'$ be a lift of $\overline{v}_i$. We define $b_{il} \in E_{l,\mathbb{Q}}(R)$ by $\sum_{l=2}^{t} b_{il}e_l = v_i$. It suffices to show that there exists a solution $\{a_{il}\}$ $(i \in \mathbb{Z}/u\mathbb{Z},\ 2 \le l \le t)$ satisfying

$$F^{\alpha_i}\dot{F}^{\beta_i}y_i - y_{i+1} \equiv \sum_{l=2}^{t} b_{il}e_l \qquad (\mathrm{mod}\ \ I_{R,\mu}M(\xi')_R). \tag{15}$$

Comparing the coefficients of e_l of the both sides of (15), we obtain

$$a_{i,l}^{\sigma^{\alpha_i+\beta_i}}\theta_l^{n_l\alpha_i - m_l\beta_i} - a_{i+1,l} \equiv b_{il} \qquad (\mathrm{mod}\ \ I_{R,\mu}E_l(R)). \tag{16}$$

for $i \in \mathbb{Z}/u\mathbb{Z}$. Since l is the same in each equation, it suffices to solve the simultaneous equations for each l. Writing a_i, b_i, n, m and θ for a_{il}, b_{il}, n_l, m_l and θ_l respectively, we have

$$a_1^{\sigma^{\sum_{i=1}^{u}(\alpha_i+\beta_i)}}\theta^{\sum_{i=1}^{u}(n\alpha_i - m\beta_i)} - a_1 \equiv r \qquad (\mathrm{mod}\ \ I_{R,\mu}E_l(R)) \tag{17}$$

for some $r \in E_{l,\mathbb{Q}}(R)$. It suffices to show that this has a solution $a_1 \in E_{l,\mathbb{Q}}(R)$ for a finite cover $\mathrm{Spec}(R)$ of S; then we get a required solution $\{a_i\}_{i=1}^{u}$ from (16).

Write $z := a_1$ and $\varrho := \sigma^{\sum_{i=1}^{u}(\alpha_i+\beta_i)}$. Note $\varrho \ne 1$ by $\alpha_i, \beta_i > 0$. We also put $\epsilon := \sum_{i=1}^{u}(n\alpha_i - m\beta_i)$. Then (17) is written as $z^{\varrho}\theta^{\epsilon} - z \equiv r$ $(\mathrm{mod}\ \ I_{R,\mu}E_l(R))$. If $\epsilon > 0$, we have a solution $z = \sum_{\ell=0}^{\infty}(-r)^{\varrho^{\ell}}\theta^{\ell\epsilon}$. Also if $\epsilon < 0$, let c be a sufficient large integer such that $\theta^{-c\epsilon} \in I_{R,\mu}E_l(R)$, and we replace R by R' so that $(R')^{p^c} = R$; then we have a solution $z = \sum_{\ell=1}^{c-1} r^{\varrho^{-\ell}}\theta^{-\ell\epsilon}$. Finally we consider the case $\epsilon = 0$. Write $z = \sum_{i=0}^{m+n-1} z_i\theta^i$ and $r = \sum_{i=0}^{m+n-1} r_i\theta^i$ with $z_i, r_i \in W_{\mathbb{Q}}(R)$. It suffices to solve $z_i^{\varrho} - z_i \equiv r_i$ $(\mathrm{mod}\ \ I_{R,\mu})$ for each $0 \le i < m+n$. Let ν_i be the biggest non-negative integer ν such that $r_i \in p^{\nu}W(R)$. We replace R with R' so that we have $(R')^{p^{\nu}} = R$. Then there exist elements t_j of R for all integers $j \ge \nu_i$ such that $z_i = \sum_{j=\nu_i}^{\infty} {}^{V^j}[t_j]$ is a solution. Indeed, putting $z_{ij} := \sum_{j'<j} {}^{V^{j'}}[t_{j'}]$, we can find $t_{j'}$ successively so that $z_{ij}^{\varrho} - z_{ij} \equiv r_i$ $(\mathrm{mod}\, I_{R,j})$. Let $j \ge \nu_i$ and suppose that we have already got such $t_{j'}$ for $j' < j$. Since $\varrho \ne 1$, after replacing R by a finite cover R', there exists a solution $t_j \in R$ of the Artin-Schreier equation $t_j^{\varrho} - t_j = ({}^{V^{-j}}(r_i - z_{ij}^{\varrho} + z_{ij}) \bmod I_R)$. Then clearly $z_i := \sum_{j=\nu_i}^{\mu-1} {}^{V^j}[t_j]$ is a solution of $z_i^{\varrho} - z_i \equiv r_i$ $(\mathrm{mod}\ \ I_{R,\mu})$. □

7 Proof of Proposition 5.2

Let $w \in {}^{J}\mathrm{W}$. Let (m_1, n_1) be the first segment of $\xi(w)$. By the definition of $\xi(w)$, there exists a p-divisible group X over an algebraically closed field k of characteristic p such that $X[p]$ is of type w and the Newton polygon of X is $\xi(w)$. Write $M = \mathbb{D}(X)$. Choose an embedding $\imath : M \to M(\xi(w))$ and let $\jmath : M(\xi(w)) \to M_{m_1,n_1}$ be the natural projection. Put $M_1 = \jmath \circ \imath(M)$. Let $f_0 : X \to X_1$ be the homomorphism of p-divisible groups corresponding to $M \to M_1$. Let X'_0 be the kernel of f_0. Note X'_0 is a p-divisible group. Thus we have an exact sequence of p-divisible groups

$$0 \longrightarrow X'_0 \longrightarrow X \xrightarrow{f_0} X_1 \longrightarrow 0. \tag{18}$$

Lemma 7.1. *X_1 is minimal, i.e., $X_1 \simeq H_{m_1,n_1}$.*

Proof. Recall [4], Corollary 5.4, whose dual is as follows. Let λ_v be the optimal lower bound of the first Newton slopes of p-divisible groups with p-kernel type v for each $v \in {}^{\mathrm{J}}\mathrm{W}$; then we have

$$\lambda_v = \min\{m/(m+n) \mid G_{v,\Omega} \overset{\exists}{\twoheadrightarrow} H_{m,n}[p]_\Omega \text{ for some alg. closed field } \Omega\}. \tag{19}$$

Note that λ_v is equal to the first slope of $\xi(v)$.

Let w and w_1 be the final types of $X[p]$ and $X_1[p]$ respectively. Since $X[p] \twoheadrightarrow X_1[p]$, i.e., $G_{w,k} \twoheadrightarrow G_{w_1,k}$, we have $\lambda_w \leq \lambda_{w_1}$ by (19). By the construction of X_1, the (first) Newton slope of X_1 is λ_w; hence we have $\lambda_w \geq \lambda_{w_1}$. Thus $\lambda_w = \lambda_{w_1}$. Then (19) implies that there exists a surjective homomorphism $H_{m_1,n_1}[p]_\Omega \twoheadrightarrow G_{w_1,\Omega}$ for some $\Omega = \overline{\Omega}$. This is an isomorphism, since $H_{m_1,n_1}[p]$ and G_{w_1} have the same rank ($= m_1 + n_1$). $\square$

We use induction on the rank of w to prove Proposition 5.2. Assume that w is not minimal. It suffices to show the case that

(*) G_w has no direct factor which is isomorphic to $H_{m_1,n_1}[p]$.

Indeed if $G_w = G_v \oplus H_{m_1,n_1}[p]$, then v is not minimal and our problem can be reduced to the case v. Hence we assume (*) from now on. Let Z and Z_1 be F-zips of $X[p]$ and $X_1[p]$ respectively. Let $\mathcal{B}$ and $\mathcal{B}_1$ be the final types of Z and Z_1 respectively. Now (*) implies that $\Omega_\infty(\mathcal{B},\mathcal{B}_1) = \emptyset$. Consider the space $\Sigma := \mathrm{Hom}(Z, Z_1)$, which is isomorphic to $\prod_{\omega\in\Omega_o(\mathcal{B},\mathcal{B}_1)} \mathbb{K}_\omega$; hence Σ is irreducible. Let f be the universal homomorphism $Z_\Sigma \to (Z_1)_\Sigma$.

Lemma 7.2. *Let $T \to \Sigma$ be any dominant morphism of k-schemes. Then f_T is not "constant up to $\mathrm{Aut}(Z_T)$". Here we say that f_T is constant up to $\mathrm{Aut}(Z_T)$ if there exists a section $x = \mathrm{Spec}(k) \to T$ such that $f_T = (f_x)_T \circ \kappa$ for an automorphism κ of Z_T.*

Proof. Let $x = \mathrm{Spec}(k) \to T$ be any section and κ any automorphism of Z_T. Let i be the largest integer such that $\mathrm{Fil}^i \mathrm{Hom}(Z, Z_1) = \mathrm{Hom}(Z, Z_1)$. Since $\Omega(\mathcal{B},\mathcal{B}_1)$ consists of finite slices, we have $\dim \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1) < \dim \mathrm{Hom}(Z, Z_1)$. We write $\kappa = \kappa_o + \kappa_\infty$ with $\kappa_o \in \mathrm{End}(Z)_o(T)$ and $\kappa_\infty \in \mathrm{End}(Z)_\infty(T)$. It follows from Corollary 3.4 that $(f_x)_T \circ \kappa$ is in $\mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1)(T) + (f_x)_T \circ \kappa_\infty$. Since $\mathrm{End}(Z)_\infty$ is discrete, κ_∞ factors through $\mathrm{Spec}(k)$. Hence the dimension of the scheme-theoretic image of $(f_x)_T \circ \kappa : T \to \mathrm{Hom}(Z, Z_1)$ is less than or equal to $\dim \mathrm{Fil}^{i+1} \mathrm{Hom}(Z, Z_1)$. On the other hand, the morphism $f_T : T \to \mathrm{Hom}(Z, Z_1)$ is dominant. Hence we have $f_T \neq (f_x)_T \circ \kappa$. $\square$

Let η denote the generic point of Σ and let w' be the type of the kernel of f_η. Let U be the open subvariety of Σ consisting of $u \in U$ such that f_u is surjective and the kernel of f_u is of type w'. Choose a finite surjective morphism $S \to U$ which trivializes $Z' := \ker \rho_U$, i.e., $Z'_S \simeq (Z_{w'})_S$.

$$0 \longrightarrow (Z_{w'})_S \longrightarrow Z_S \xrightarrow{f_S} (Z_1)_S \longrightarrow 0. \tag{20}$$

By Corollary 2.2, there exists a display M' over k such that $M'/IM' \simeq Z_{w'}$ and the Newton polygon of M' is $\xi(w')$. Choose an isogeny $M(\xi(w')) \to M'$. Put

$$\zeta := \xi(w') + (m_1, n_1). \tag{21}$$

Applying the result of §6 to (20), for a finite surjective morphism $\mathrm{Spec}(R) \to S$, we obtain an isogeny

$$\rho : \quad M(\zeta)_R \longrightarrow \mathcal{M} \tag{22}$$

with $\phi : \mathcal{M} \to (M_1)_R$ and $\theta : \mathcal{M}/I_R\mathcal{M} \simeq Z_R$ satisfying the commutative diagrams (9) and (10).

Lemma 7.3. *We have $\zeta = \xi(w)$.*

Proof. Since $\mathcal{M}$ has Newton polygon ζ and p-kernel type w, we have $\zeta \prec \xi(w)$ by the definition of $\xi(w)$. Let w'_0 be the type of $X'_0[p]$. Since $w'_0 \subset w'$, we have $\xi(w'_0) \prec \xi(w')$ by Corollary 4.2. Note the Newton polygon $\xi(w)$ of X is equal to $\xi(w'_0) + (m_1, n_1)$. This is less than or equal to $\xi(w') + (m_1, n_1) = \zeta$. □

It remains to show

Lemma 7.4. *ρ is not constant.*

Proof. Applying Lemma 7.2 to $T := \mathrm{Spec}(R) \to \Sigma$, we have that f_R is not constant up to $\mathrm{Aut}(Z_R)$. It follows from the diagram (10) that $\overline{\phi}$ is non-constant; hence so is ϕ. Then we have the lemma by the diagram (9). □

References

[1] N. Bourbaki: *Lie groups and Lie algebras. Chapters 4–6.* Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[2] T. Ekedahl and G. van der Geer: *Cycle classes of the E-O stratification on the moduli of abelian varieties.* Algebra, arithmetic and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., **269**, Birkhäuser, Boston, 2009; pp. 567–636.

[3] S. Harashita: *Ekedahl-Oort strata and the first Newton slope strata.* J. Algebraic Geom. **16** (2007), no. 1, 171–199.

[4] S. Harashita: *Successive extensions of minimal p-divisible groups.* J. Pure Appl. Algebra **213** (2009), 1823-1834.

[5] S. Harashita: *Configuration of the central streams in the moduli of abelian varieties.* Asian J. Math. **13** (2009), no. 2, 215–250.

[6] S. Harashita: *Generic Newton polygons of Ekedahl-Oort strata: Oort's conjecture.* Ann. Inst. Fourier (Grenoble) **60** (2010), no. 5, 1787–1830.

[7] X. He: *The G-stable pieces of the wonderful compactification.* Trans. Amer. Math. Soc. **359** (2007), no. 7, 3005–3024.

[8] N. M. Katz: *Slope filtration of F-crystals.* Géom. Alg. Rennes, Vol. I, Astérisque **63** (1979), Soc. Math. France, pp.113-164.

[9] H. Kraft: *Kommutative algebraische p-Gruppen (mit Anwendungen auf p-divisible Gruppen und abelsche Varietäten).* Sonderforschungsbereich. Bonn, September 1975. Ms. 86 pp.

[10] E. Lau: *Displays and formal p-divisible groups.* Invent. Math. **171** (2008), 617–628.

[11] Yu. I. Manin, *Theory of commutative formal groups over fields of finite characteristic.* Uspehi Mat. Nauk **18** no. 6 (114) (1963), pp. 3-90.

[12] B. Moonen: *Group schemes with additional structures and Weyl group cosets.* Moduli of abelian varieties (Ed. C. Faber, G. van der Geer, F. Oort), Progr. Math., **195**, Birkhäuser, Basel, 2001; pp. 255–298.

[13] B. Moonen and T. Wedhorn: *Discrete invariants of varieties in positive characteristic.* Int. Math. Res. Not. 2004, no. **72**, 3855–3903.

[14] F. Oort: *Simple p-kernels of p-divisible groups.* Adv. Math. **198** (2005), no. 1, 275–310.

[15] F. Oort: *Minimal p-divisible groups.* Ann. of Math. (2) **161** (2005), no. 2, 1021–1036.

[16] F. Oort: *Foliations in moduli spaces of abelian varieties.* J. Amer. Math. Soc. **17** (2004), no. 2, 267–296.

[17] F. Oort: *A stratification of a moduli space of abelian varieties.* Progr. Math., **195**, pp. 345–416, Birkhäuser, Basel, 2001.

[18] M. Rapoport and Th. Zink: *Period spaces for p-divisible groups.* Annals of Mathematics Studies **141** (1996), Princeton University Press.

[19] A. Vasiu: *Level m stratifications of versal deformations of p-divisible groups.* J. Algebraic Geom. **17** (2008), no. 4, 599–641.

[20] A. Vasiu: *Reconstructing p-divisible groups from their truncations of small level.* Comment. Math. Helv. **85** (2010), no. 1, 165–202.

[21] E. Viehmann: *Truncations of level 1 of elements in the loop group of a reductive group.* arXiv:0907.2331v1

[22] T. Wedhorn: *Specialization of F-zips.* arXiv: 0507175v1

[23] Th. Zink: *The display of a formal p-divisible group.* Cohomologies p-adiques et applications arithmétiques, I. Astérisque No. **278** (2002), 127–248.