Successive extensions of minimal p-divisible groups

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Abstract

We study truncated Barsotti-Tate groups of level one (BT_1) and their extensions to p-divisible groups. Firstly we show that any BT_1 contains a certain minimal BT_1 as a non-zero subgroup scheme. This proves that any BT_1 is written as a successive extension of minimal BT_1's. Secondly we prove that any successive extension of minimal BT_1's which is a BT_1 can be extended to a certain successive extension of minimal p-divisible groups. As an application, we determine the optimal upper bound of the last Newton slopes of p-divisible groups with given isomorphism type of p-kernel.

1 Introduction

We fix throughout a prime number p. Let k be an algebraically closed field of characteristic p. In this paper, we present a new method to extend truncated Barsotti-Tate groups of level one (BT_1) over k to p-divisible groups over k. A good point of this method is that we can extend a BT_1 to p-divisible groups with various determinable isogeny types (Newton polygon).

Kraft and Oort classified the isomorphism classes of BT_1's over k by *final sequences* (cf. §2.3). For non-negative integers c, d with $\gcd(c, d) = 1$, let $H_{c,d}$ denote the simple minimal p-divisible group of slope $c/(c + d)$ (cf. §3.1). In §4.1 we associate to any final sequence ν non-negative integers m_ν, n_ν and a natural number e_ν in a combinatorial way. The first aim of this paper is to prove

Theorem 1.1. *Let G be a BT_1 over k with final sequence ν. Then there exists an injective homomorphism over k*

$$H_{m,n}^{\oplus e}[p] \longrightarrow G$$

with $m = m_\nu$, $n = n_\nu$ and $e = e_\nu$.

This shows that any BT_1 can be written as a successive extension of minimal BT_1's (Definition 3.2), see Lemma 2.2. The second aim is to prove

Theorem 1.2. *Let G be a BT_1. If G is a successive extension of $\{H_{m_i,n_i}[p]\}_{i=1}^t$, then G can be extended to a certain successive extension of minimal p-divisible groups $\{H_{m_i,n_i}\}_{i=1}^t$.*

Hence for any pair of final sequence ν and Newton polygon $\xi = \sum_{i=1}^t (m_i, n_i)$ such that a BT_1 having final sequence ν is written as a successive extension of $\{H_{m_i,n_i}[p]\}_{i=1}^t$, there exists a p-divisible group with final sequence ν and Newton polygon ξ. Note that for given ν such ξ is not unique in general.

As an application, we shall show

Corollary 1.3. *The optimal upper bound of the last Newton slopes of p-divisible groups with given final sequence ν is equal to $\rho_\nu = m_\nu/(m_\nu + n_\nu)$ (see Definition 4.1).*

This is an unpolarized analogue of [3], Theorem 4.1. An obstruction in the unpolarized case is the absence of a good moduli space like the moduli space of principally polarized abelian varieties. As an alternative method, we use the theorems above. Let us explain the background of Corollary 1.3. A final goal of our research is to solve the following classical problem:

(P) *Let ν be a final sequence. Classify the possible isogeny classes (Newton polygons) of p-divisible groups over k with final sequence ν.*

Since it seems difficult to give a complete answer to **(P)**, one may propose a weaker problem:

(P′) *Show the existence of the optimal upper bound (w.r.t. $\prec$) of the Newton polygons of p-divisible groups over k with final sequence ν, and find an algorithm determining it.*

Note **(P′)** is also open in general. However I believe that **(P′)** would be beautifully solved by induction, and expect that Corollary 1.3 would be the first step of the induction.

In the last subsection, combining Theorems 1.1, 1.2 and Corollary 1.3, we shall show

Corollary 1.4. *A* BT_1 *is* BT_1*-simple if and only if it is minimal and indecomposable.*

This has already been obtained by Oort ([10], Theorem A).

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Notations

For non-negative integers m, n we denote by $\gcd(m, n)$ the greatest common divisor, where for convenience we set $\gcd(m, 0) = \gcd(0, m) = m$ for $m \in \mathbb{Z}_{\geq 0}$. For an integral domain R, we denote by $\mathrm{frac}(R)$ its field of fractions. For a set S, we denote by $|S|$ the cardinality of S.

2 Preliminaries

2.1 The Dieudonné theory

Let K be a perfect field of characteristic p and $W(K)$ the ring of infinite Witt vectors with coordinates in K. Let A_K be the p-adic completion of the associative ring

$$W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}x - x^\sigma \mathcal{F}, \mathcal{V}x^\sigma - x\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \forall x \in W(K)) \tag{2.1.1}$$

with the Frobenius automorphism σ of $W(K)$. A *Dieudonné module* over $W(K)$ is a left A_K-module which is finitely generated as a $W(K)$-module. A Dieudonné module is called *free* if it is free as $W(K)$-module.

The covariant Dieudonné theory says that there is a canonical categorical equivalence $\mathbb{D}$ from the category of p-torsion finite commutative group schemes (resp. p-divisible groups) over K to the category of Dieudonné modules over $W(K)$ which are of finite length (resp. free). We write F and V for "Frobenius" and "Verschiebung" on commutative group schemes. The covariant Dieudonné functor $\mathbb{D}$ satisfies $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$.

2.2 The Dieudonné-Manin classification

A *segment* is a pair (m, n) of non-negative integers with $\gcd(m, n) = 1$. The *slope* $\lambda(s)$ of a segment $s = (m, n)$ is defined to be $m/(m + n)$.

For a sequence $(s_1, \ldots, s_t)$ of segments $s_i = (m_i, n_i)$, putting $P_j := (\sum_{i=1}^{j}(m_i+n_i), \sum_{i=1}^{j} m_i) \in \mathbb{R}^2$ for $0 \leq j \leq t$, let $\mathcal{L} = \mathcal{L}(s_1, \ldots, s_t)$ denote the line graph in $\mathbb{R}^2$ passing through $P_0, \ldots, P_t$ in this order. Put $h := \sum_{i=1}^{t}(m_i + n_i)$. For a point $Q \in [0, h] \times \mathbb{R}$, we say $Q \prec \mathcal{L}$ if Q is on or above $\mathcal{L}$. We say, for two line graphs $\mathcal{L}$, $\mathcal{L}'$ as above with the same end point, that $\mathcal{L}' \prec \mathcal{L}$ if $Q \prec \mathcal{L}$ for all $Q \in \mathcal{L}'$.

A *Newton polygon* is a line graph of the form $\mathcal{L}(s_1, \ldots, s_t)$ with $\lambda(s_1) \leq \cdots \leq \lambda(s_t)$. Since this Newton polygon is the biggest one with respect to $\prec$ among line graphs obtained from the set $\{s_1, \ldots, s_t\}$, we usually write this Newton polygon as $s_1 + \cdots + s_t$.

For a segment (m, n), we define a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(G_{m,n}) = A_{\mathbb{F}_p}/A_{\mathbb{F}_p}(\mathcal{F}^m - \mathcal{V}^n). \tag{2.2.1}$$

By the Dieudonné-Manin classification [6], for any p-divisible group $\mathcal{G}$ over a field K of characteristic p, there is an isogeny over an algebraically closed field containing K from $\mathcal{G}$ to

$$\bigoplus_{i=1}^{t} G_{m_i,n_i} \tag{2.2.2}$$

for some finite set of segments $s_i = (m_i, n_i)$. Thus we get a Newton polygon $s_1 + \cdots + s_t$, which is denoted by $\mathrm{NP}(\mathcal{G})$. One may suppose $\lambda(s_1) \leq \cdots \leq \lambda(s_t)$. We call $\lambda_i(\mathcal{G}) := \lambda(s_i)$ the *i-th Newton slope* of $\mathcal{G}$. We set

$$\rho_i(\mathcal{G}) = \lambda_{t+1-i}(\mathcal{G}) \tag{2.2.3}$$

and call $\rho_1(\mathcal{G})$ the *last Newton slope* (or the *highest Newton slope*). Note the height of $\mathcal{G}$ is equal to $h = \sum_{i=1}^{t}(m_i + n_i)$ and $\dim_K \mathrm{Lie}(\mathcal{G}) = \sum_{i=1}^{t} m_i$. We remark that h and $\sum_{i=1}^{t} m_i$ depend only on the p-kernel $\mathcal{G}[p]$.

2.3 The Kraft-Oort classification

We review the classification of BT_1's by Kraft, which was reobtained by Oort. The main references are [5] and [8]. Let K be a field of characteristic p. All group schemes will be over K and all homomorphisms of them will be over K.

Definition 2.1. (1) A finite commutative group scheme G is said to be a BT_1 if

$$\mathrm{Im}(V : G^{(p)} \to G) = \mathrm{Ker}(F : G \to G^{(p)}),$$
$$\mathrm{Im}(F : G \to G^{(p)}) = \mathrm{Ker}(V : G^{(p)} \to G).$$

(2) Assume K is perfect. The Dieudonné module of a BT_1 is called a *truncated Dieudonné module of level one (*DM_1*)*.

We shall use the following basic lemma.

Lemma 2.2. (1) *Let $f : G_1 \to G_2$ be a surjective homomorphism of* BT_1*'s. Then* $\mathrm{Ker}\, f$ *is also a* BT_1.

(2) *Let* $f : G_1 \to G_2$ *be an injective homomorphism of* BT_1*'s. Then* $\mathrm{Coker}\, f$ *is also a* BT_1.

Proof. (1) Set $G_3 = \mathrm{Ker}\, f$. For $1 \le i \le 3$, by $p \cdot G_i = 0$, we have the complexes

$$C_i^\bullet : \xrightarrow{F} G_i^{(p)} \xrightarrow{V} G_i \xrightarrow{F} G_i^{(p)} \xrightarrow{V} G_i \xrightarrow{F} \cdot$$

For $i = 1, 2$ we have $H^j(C_i^\bullet) = 0$ for any $j \in \mathbb{Z}$ by the definition of BT_1's. The long exact sequence deduced from the exact sequence

$$0 \longrightarrow C_3^\bullet \longrightarrow C_1^\bullet \longrightarrow C_2^\bullet \longrightarrow 0$$

shows $H^j(C_3^\bullet) = 0$ for any $j \in \mathbb{Z}$. This means that G_3 is a BT_1. Similarly we obtain (2). □

From this lemma, any direct factor of a BT_1 in the category of finite commutative group schemes is a BT_1.

We review the classification of BT_1's in terms of final sequences.

Definition 2.3. A *final sequence* of length h is a map $\nu : \{0, 1, \dots, h\} \to \{0, 1, \dots, h\}$ satisfying $\nu(0) = 0$ and $\nu(i-1) \le \nu(i) \le \nu(i-1) + 1$.

Let G be a BT_1 over K of rank p^h. For any subgroup scheme G' of G over $\overline{K}$ and for any word w of V, F^{-1}, we define $w \cdot G'$ inductively by $V \cdot G' := VG'^{(p)}$ and $F^{-1} \cdot G' := F^{-1}(G'^{(p)} \cap FG)$. Then there exists a unique final sequence ν of length h such that for any word w of V, F^{-1} we have $\nu(\mathrm{length}(w \cdot G)) = \mathrm{length}(Vw \cdot G)$. Thus we have a canonical map

$$\mathrm{FS} : \quad \{\mathrm{BT}_1 \text{ of length } d \text{ over } K\}/K\text{-isom.} \longrightarrow \{\text{final sequence of length } d\}. \qquad (2.3.1)$$

The Kraft-Oort classification is described as:

Theorem 2.4 ([5])**.** *If* K *is algebraically closed, then* FS *is bijective.*

Assume K is algebraically closed. We shall use k instead of K.

Definition 2.5. Let G be a BT_1 and ν its final sequence.

(1) We call G (or ν) BT_1-*simple* if there is no non-zero proper BT_1 subgroup scheme of G.

(2) We call G (or ν) *indecomposable* if there is no non-zero proper direct factor of G.

Definition 2.6. (1) A *final type* of length h is a pair (B, δ) consisting of a totally ordered finite set B with $|B| = h$ and a map $\delta : B \to \{0, 1\}$.

(2) Let $\mathcal{B} = (B, \delta)$ and $\mathcal{B}' = (B', \delta')$ be two final types. We say $\mathcal{B}$ and $\mathcal{B}'$ are *isomorphic* if there exists an ordered bijection f from B to B' such that $\delta = \delta' \circ f$.

For a final type (B, δ), we define an automorphism $\pi = \pi_\delta$ of B as follows. Let $B = \{b_1 < \cdots < b_h\}$ and set $B_- = \{b \in B \mid \delta(b) = 0\}$ and $B_+ = \{b \in B \mid \delta(b) = 1\}$. Put $h_0 = |B_-|$. Let π_- and π_+ be the ordered maps

$$\pi_- : \quad B_- \longrightarrow \{b_1, \dots, b_{h_0}\},$$

$$\pi_+ : \quad B_+ \longrightarrow \{b_{h_0+1}, \dots, b_h\}.$$

Then π is defined by

$$\pi(b) = \begin{cases} \pi_-(b) & \text{if} \quad b \in B_-, \\ \pi_+(b) & \text{if} \quad b \in B_+. \end{cases} \tag{2.3.2}$$

We call π_δ the *automorphism of B associated with δ.* We define an automorphism $\varpi = \varpi_\delta$ of $\{1, \ldots, h\}$ by $\pi(b_i) = b_{\varpi(i)}$ for all $1 \leq i \leq h$.

Obviously we have

Lemma 2.7. *The map from the set of final sequences of length h to the set of equivalence classes of final types of length h sending ν to the class of (B, δ) defined by $B = \{b_1 < \cdots < b_h\}$ and $\delta(b_i) = 1 - \nu(i) + \nu(i-1)$ for all $1 \leq i \leq h$ is bijective.*

Let G' and G'' be BT_1's. Let ν' and ν'' be their final sequences and let $\mathcal{B}'$ and $\mathcal{B}''$ be their final types. We shall write the final sequence of $G' \oplus G''$ as $\nu' \oplus \nu''$ and its final type as $\mathcal{B}' \oplus \mathcal{B}''$. Also we write a homomorphism $f : G' \to G''$ as $f : \nu' \to \nu''$, etc.

We review the inverse map of FS. Let ν be a final sequence and $\mathcal{B} = (B, \delta)$ be its final type with $B = \{b_1 < \cdots < b_h\}$. We can construct a BT_1, say G, having final sequence ν as follows. The Dieudonné module of G is defined to be

$$\mathbb{D}(G) = \bigoplus_{i=1}^{h} kZ_i \tag{2.3.3}$$

with $\mathcal{F}$ and $\mathcal{V}$ operations

$$\mathcal{F}Z_i = \begin{cases} Z_{\varpi(i)} & \text{if } \delta(b_i) = 0, \\ 0 & \text{if } \delta(b_i) = 1 \end{cases} \quad \text{and} \quad \mathcal{V}Z_{\varpi(i)} = \begin{cases} Z_i & \text{if } \delta(b_i) = 1, \\ 0 & \text{if } \delta(b_i) = 0. \end{cases}$$

By this construction, from any π-stable subset B' of B we obtain a direct factor G' of G, which is defined by $\mathbb{D}(G') = \bigoplus_{b_i \in B'} kZ_i$.

3 Minimal p-divisible groups

In this subsection we review the definition of minimal p-divisible groups (cf. [2], §5.3 and [9]) and show some basic facts.

3.1 Definitions

Definition 3.1. For non-negative integers m, n with $m + n > 0$, we define a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p x_i$$

with $\mathcal{F}, \mathcal{V}$ operations:

$$\mathcal{F}x_i = x_{i+n} \quad \text{and} \quad \mathcal{V}x_i = x_{i+m} \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \tag{3.1.1}$$

where x_i $(i \in \mathbb{Z}_{\geq m+n})$ are defined as satisfying $x_{i+m+n} = px_i$ for $i \in \mathbb{Z}_{\geq 0}$.

Note $\mathbb{D}(H_{m,n})$ has an endomorphism ϑ defined by $\vartheta(x_i) = x_{i+1}$. For an arbitrary perfect field K, the Dieudonné module $\mathbb{D}(H_{m,n} \otimes K)$ has a $W(K)$-basis $\{x_0, \dots, x_{m+n-1}\}$ satisfying the equations (3.1.1), which is called a *minimal basis* of $\mathbb{D}(H_{m,n} \otimes K)$. We call x_0 the *highest element* of the minimal basis. We have $H_{m,n}^{\oplus e} \simeq H_{em,en}$ for any $e \in \mathbb{Z}_{\geq 1}$.

For a Newton polygon $\xi = \sum_{i=1}^{t}(m_i, n_i)$, we denote by $H(\xi)$ the p-divisible group

$$\bigoplus_{i=1}^{t} H_{m_i,n_i}. \tag{3.1.2}$$

Note the Newton polygon of $H(\xi)$ is equal to ξ.

Definition 3.2. Let $\mathcal{G}$ be a p-divisible group and let G be a BT_1. We call $\mathcal{G}$ (resp. G) *minimal* if there exist a Newton polygon ξ and an isomorphism from $\mathcal{G}$ (resp. G) to $H(\xi)$ (resp. $H(\xi)[p]$) over an algebraically closed field.

3.2 A description of $\mathbb{D}(H_{m,n})$

In this subsection we assume $m, n > 0$ and $\gcd(m, n) = 1$, and put $M = \mathbb{D}(H_{m,n})$. Set

$$u = \min\{m, n\} \quad \text{and} \quad v = \max\{m, n\}. \tag{3.2.1}$$

Let $\{x_0, \dots, x_{m+n-1}\}$ be a minimal basis of M. For any x_i with $i < u$, let $\alpha(x_i)$ denote the least integer α such that $\mathcal{F}^{\alpha+1}(x_i) \in pM$. Note $\alpha(x_i) \geq 1$ for all $i < u$. For any x_i with $i \geq v$, let $\beta(x_i)$ denote the largest integer β such that $x_i \in \mathcal{V}^{\beta}M$. Note $\beta(x_i) \geq 1$ for all $i \geq v$.

Set $X_0 = x_0$. We define inductively α_i, $\beta_{i+1} \in \mathbb{Z}_{\geq 1}$ and $X_{i+1} \in \{x_0, \dots, x_u\}$ for $i \in \mathbb{Z}_{\geq 0}$ by $\alpha_i = \alpha(X_i)$ and $\beta_{i+1} = \beta(\mathcal{F}^{\alpha_i}(X_i))$, and $\mathcal{V}^{\beta_{i+1}}X_{i+1} = \mathcal{F}^{\alpha_i}X_i$. Note $X_{i+u} = X_i$ for any $i \geq 0$. Thus we get α_i, β_i and X_i for $i \in \mathbb{Z}/u\mathbb{Z}$. Obviously we have

Lemma 3.3. *Suppose $m, n > 0$ and $\gcd(m, n) = 1$. Then putting*

$$I_i = \mathcal{F}^{\alpha_i}X_i - \mathcal{V}^{\beta_{i+1}}X_{i+1},$$

we have

$$\mathbb{D}(H_{m,n} \otimes K) = A_K\langle X_1, \dots, X_u\rangle / A_K\langle I_1, \dots, I_u\rangle,$$

for any perfect field K of characteristic p.

3.3 The final type of $H_{m,n}[p]$

We recall a part of [4], §4.5. Let m, n be non-negative integers with $m + n > 0$. Let $\nu_{m,n}$ be the final sequence of $H_{m,n}[p]$. Then we have $\nu_{m,n} = (0, \dots, 0, 1, \dots, m)$ with n zeros ([4], Lemma 4.13), and the final type $\mathcal{B}_{m,n} = (B_{m,n}, \delta_{m,n})$ of $\nu_{m,n}$ is given by $B_{m,n} = \{b_1 < \cdots < b_{m+n}\}$ and by $\delta_{m,n}(b_i) = 1$ for $1 \leq i \leq n$ and $\delta_{m,n}(b_i) = 0$ for $n < i \leq m+n$. Let $\pi_{m,n}$ be the automorphism of $B_{m,n}$ associated with $\delta_{m,n}$. Then we have the commutative diagram

$$\begin{array}{ccc} B_{m,n} & \longrightarrow & \mathbb{Z}/(m+n)\mathbb{Z} \\ {\scriptstyle \pi_{m,n}}\big\downarrow & & \big\downarrow{\scriptstyle -n} \\ B_{m,n} & \longrightarrow & \mathbb{Z}/(m+n)\mathbb{Z}, \end{array} \tag{3.3.1}$$

where the horizontal maps send b_i to the class of $i - 1$. Note $H_{m,n}[p]$ is indecomposable if and only if $\gcd(m, n) = 1$ (cf. [4], Definition 4.3 and Corollary 4.15).

4 Construction of embeddings of certain minimal BT_1's

The aim of this section is to prove Theorem 1.1. Before that, we define m_ν, n_ν, e_ν and so on, and investigate some basic properties of them.

4.1 Ψ-cycles

Let ν be a final sequence, and let $\mathcal{B} = (B, \delta)$ be its final type with $B = \{b_1 < \cdots < b_h\}$. We define a map

$$\tilde{\Psi}: \quad B \longrightarrow B$$

by sending b_i to

$$\begin{cases} b_{\nu(i)} & \text{if } \nu(i) \neq 0, \\ b_{\nu(h)+i} & \text{if } \nu(i) = 0. \end{cases} \tag{4.1.1}$$

We get a non-empty ordered subset of B:

$$\mathcal{D} = \mathcal{D}_\nu := \bigcap_{j=1}^{\infty} \operatorname{Im} \tilde{\Psi}^j, \tag{4.1.2}$$

where $\tilde{\Psi}^j$ denotes the composition of j copies of $\tilde{\Psi}$. Then $\tilde{\Psi}$ induces an automorphism

$$\Psi: \quad \mathcal{D} \longrightarrow \mathcal{D}. \tag{4.1.3}$$

Set

$$\mathcal{C} = \mathcal{C}_\nu := \mathcal{D} \cap \{b_{\nu(h)+1}, b_{\nu(h)+2}, \ldots, b_h\}. \tag{4.1.4}$$

Since for any $i \leq i'$ we have

$$\nu(i) \leq \nu(i') \quad \text{and} \quad \nu(h) + i - \nu(i) \leq \nu(h) + i' - \nu(i'), \tag{4.1.5}$$

we have the commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathbb{Z}/|\mathcal{D}|\mathbb{Z} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle -|\mathcal{C}|} \\ \mathcal{D} & \longrightarrow & \mathbb{Z}/|\mathcal{D}|\mathbb{Z}, \end{array} \tag{4.1.6}$$

where the horizontal maps are defined by the unique ordered map from $\mathcal{D}$ to $\{0, \cdots, |\mathcal{D}| - 1\}$. We put $e_\nu := \gcd(|\mathcal{C}|, |\mathcal{D}|)$ and define $m_\nu, n_\nu \in \mathbb{Z}$ by $|\mathcal{C}| = e_\nu n_\nu$ and $|\mathcal{D}| = e_\nu(m_\nu + n_\nu)$.

Definition 4.1. (1) The *slope associated with* ν is the rational number

$$\rho_\nu = m_\nu/(m_\nu + n_\nu).$$

(2) We call the natural number e_ν the *Ψ-multiplicity in* ν.

Proposition 4.2. *Let $\mathcal{G}$ be a p-divisible group with $\mathrm{FS}(\mathcal{G}[p]) = \nu$. Then the last Newton slope $\rho_1(\mathcal{G})$ of $\mathcal{G}$ is less than or equal to ρ_ν.*

Proof. Let $\mathfrak{s}$ be the slope function of $\mathcal{G}$ (cf. [1], IV. 5). Recall that $\mathfrak{s}$ is the continuous real-valued function on $\mathbb{R}$ defined so that for each $\lambda \in \mathbb{R}$, the straight line with slope λ tangent to the Newton polygon of $\mathcal{G}$ passes through the point $(h, \mathfrak{s}(\lambda))$. Hence in order to prove the proposition, it suffices to show $\mathfrak{s}(\rho_\nu) = \dim \mathrm{Lie}(\mathcal{G})$.

Set $m = m_\nu$ and $n = n_\nu$. By the same argument as in [3], Proposition 6.1, we have

$$V^{1+\alpha(m+n)} \cdot \mathcal{G} \subset p^{\alpha n}\mathcal{G}$$

for any $\alpha \in \mathbb{Z}_{\geq 0}$. Hence for any $\varepsilon \in \mathbb{Q}_{>0}$ and for any $\beta \in \mathbb{N}$ with $\beta\varepsilon \in \mathbb{Z}_{\geq 1}$, we obtain $\mathcal{F}^{\beta(m+n+\varepsilon)}M \subset p^{\beta n}M$, which can be paraphrased as

$$p^{\beta(m+\varepsilon)}M \subset \mathcal{V}^{\beta(m+n+\varepsilon)}M. \tag{4.1.7}$$

By [1], Corollary on p. 88, the slope function $\mathfrak{s}$ has the property:

$$\mathfrak{s}\left(\frac{m+\varepsilon}{m+n+\varepsilon}\right) = \lim_{\beta\to\infty} \frac{\mathrm{length}(M/(\mathcal{V}^{\beta(m+n+\varepsilon)}M + p^{\beta(m+\varepsilon)}M))}{\beta(m+n+\varepsilon)}.$$

From (4.1.7), this is equal to

$$\lim_{\beta\to\infty} \frac{\mathrm{length}(M/(\mathcal{V}^{\beta(m+n+\varepsilon)}M))}{\beta(m+n+\varepsilon)} = \dim \mathrm{Lie}(\mathcal{G}).$$

The continuity of $\mathfrak{s}$ implies $\mathfrak{s}(\rho_\nu) = \dim \mathrm{Lie}(\mathcal{G})$. □

Remark 4.3. Apology: in the proof of [3], Proposition 6.3, the author confused the contravariant theory [1] and the covariant theory. But anyway [3], Proposition 6.3 is true for any quasi-polarized p-divisible group.

4.2 Interrelations between π-cycles and Ψ-cycles

Let $\nu, \mathcal{B}, \mathcal{D}$ and Ψ be as in the previous subsection. Write $\mathcal{B} = (B, \delta)$ and $B = \{b_1 < \cdots < b_h\}$.

Definition 4.4. We define a final type $(\mathcal{D}, \gamma)$ by

$$\gamma(b) = \begin{cases} 1 & \text{if} \quad \nu(i) = 0, \\ 0 & \text{otherwise} \end{cases}$$

for $b = b_i \in \mathcal{D}$.

Set $m = m_\nu$, $n = n_\nu$ and $e = e_\nu$. Write $\mathcal{D} = \{c_1, \dots, c_{e(m+n)}\}$. Clearly we have

$$\gamma(c_i) = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq en, \\ 0 & \text{otherwise}, \end{cases}$$

and the automorphism π_γ of $\mathcal{D}$ associated with γ is equal to Ψ. Hence there is an isomorphism of final types

$$\kappa: \quad \mathcal{B}_{m,n}^{\oplus e} \xrightarrow{\sim} (\mathcal{D}, \gamma). \tag{4.2.1}$$

Let us investigate the interrelation between π on B and Ψ on $\mathcal{D}$.

Lemma 4.5. *Let b be an element of $\mathcal{D}$.*

(1) *We have $\delta(b) \geq \gamma(b)$ and $\pi(b) \geq \Psi(b)$.*

(2) *$\delta(b) = \gamma(b)$ if and only if $\pi(b) = \Psi(b)$.*

Proof. Write $b = b_i$. First note

$$\gamma(b) = 1 \iff \nu(i) = 0 \implies \nu(i-1) = \nu(i) \iff \delta(b) = 1. \tag{4.2.2}$$

(1) If $\delta(b) < \gamma(b)$ held, then $\delta(b) = 0$ and $\gamma(b) = 1$, which contradicts with (4.2.2). Thus we have $\delta(b) \geq \gamma(b)$.

Let us show $\pi(b) \geq \Psi(b)$. **Case** $\gamma(b) = 1$**:** By $\delta(b) \geq \gamma(b)$ we have $\delta(b) = 1$. By $\gamma(b) = 1$ we have $\Psi(b) = b_{\nu(h)+i-\nu(i)}$. By $\delta(b) = 1$ we have $\pi(b) = b_{\nu(h)+i-\nu(i)}$. Hence we have $\pi(b) = \Psi(b)$. **Case** $\gamma(b) = 0$**:** We have $\Psi(b) = b_{\nu(i)}$. Note $\pi(b) = b_{\nu(i)}$ or $b_{\nu(h)+i-\nu(i)}$. Since $\nu(i) \leq i \leq \nu(h) + i - \nu(i)$, we have $\pi(b) \geq \Psi(b)$.

(2) Assume $\delta(b) = \gamma(b)$. In the case of $\delta(b) = \gamma(b) = 1$, we have seen $\pi(b) = \Psi(b)$ in the proof of (1). If $\delta(b) = \gamma(b) = 0$, then $\nu(i-1) < \nu(i) \neq 0$; hence we have $\pi(b) = b_{\nu(i)} = \Psi(b)$.

Conversely we assume $\pi(b) = \Psi(b)$. If $\delta(b) > \gamma(b)$ held, then we have $\delta(b) = 1$ and $\gamma(b) = 0$. By $\delta(b) = 1$, we have $b \leq \pi(b)$. From $\delta(b) = 1$, we have $\nu(i-1) = \nu(i)$; hence $\nu(i) < i$. From $\gamma(b) = 0$, we have $\nu(i) \neq 0$; hence $\Psi(b) = b_{\nu(i)}$. Thus $\Psi(b) = b_{\nu(i)} < b_i = b \leq \pi(b)$. This is a contradiction. $\square$

Lemma 4.6. *Let b be an element of $\mathcal{D}$, and let c be an element of B.*

(1) *If $\pi(c) \leq \Psi(b)$ and $\delta(c) = \gamma(b)$, then $c \leq b$.*

(2) *If $c \leq b$, then $\delta(\pi^{-1}(c)) \leq \gamma(\Psi^{-1}(b))$.*

Proof. (1) Assume $\pi(c) \leq \Psi(b)$ and $\delta(c) = \gamma(b)$. Let $b = b_i$ and $c = b_j$. Suppose $c > b$ held. Then we have $j > i$. If $\delta(c) = \gamma(b) = 0$, then $\pi(c) = b_{\nu(j)} \geq b_{\nu(i)} = \Psi(b)$. By the assumption $\pi(c) \leq \Psi(b)$, we have $b_{\nu(j)} = b_{\nu(i)}$. This implies $b_{\nu(j)} = b_{\nu(j-1)}$; hence we have $\delta(c) = 1$; this is a contradiction. If $\delta(c) = \gamma(b) = 1$, then $\pi(c) = b_{\nu(h)+j-\nu(j)} \geq b_{\nu(h)+i-\nu(i)} = \Psi(b)$. By the assumption $\pi(c) \leq \Psi(b)$, we have $b_{\nu(h)+j-\nu(j)} = b_{\nu(h)+i-\nu(i)}$. This implies $\nu(h) + j - \nu(j) = \nu(h) + (j-1) - \nu(j-1)$, namely $\nu(j) = \nu(j-1) + 1$; hence we have $\delta(c) = 0$; this is a contradiction. Thus $c \leq b$ has to hold.

(2) Assume $c \leq b$. It suffices to show that $\gamma(\Psi^{-1}(b)) = 0$ implies $\delta(\pi^{-1}(c)) = 0$. Suppose $\gamma(\Psi^{-1}(b)) = 0$. Write $b_i = \Psi^{-1}(b)$. Then we have $b = \Psi(b_i) = b_{\nu(i)} \leq b_{\nu(h)}$. By the assumption $c \leq b$, we have $c \leq b_{\nu(h)}$. Note $\delta(\pi^{-1}(c)) = 0$ if and only if $c \leq b_{\nu(h)}$. $\square$

We define two $\mathbb{Q}$-valued functions on B by

$$\begin{cases} \tau^+(b) = \tau^+_\nu(b) := \sum_{i=0}^\infty \delta(\pi^i(b))2^{-i}, \\ \tau^-(b) = \tau^-_\nu(b) := \sum_{i=1}^\infty \delta(\pi^{-i}(b))2^{-i} \end{cases} \quad \text{for } b \in B \tag{4.2.3}$$

and two $\mathbb{Q}$-valued functions on $\mathcal{D}$ by

$$\begin{cases} \chi^+(b) = \chi^+_\nu(b) := \sum_{i=0}^\infty \gamma(\Psi^i(b))2^{-i}, \\ \chi^-(b) = \chi^-_\nu(b) := \sum_{i=1}^\infty \gamma(\Psi^{-i}(b))2^{-i} \end{cases} \quad \text{for } b \in \mathcal{D}. \tag{4.2.4}$$

(Let μ be the final sequence of $(\mathcal{D}, \gamma)$. By definition we have $\chi^\pm_\nu(b) = \tau^\pm_\mu(b)$ for all $b \in \mathcal{D}$.)

Proposition 4.7. *Let $b \in \mathcal{D}$.*

(1) *We have $\tau^+(b) \geq \chi^+(b)$ and $\tau^-(b) \leq \chi^-(b)$.*

(2) *The following three conditions are equivalent:*

 (i) $\tau^+(b) = \chi^+(b)$;

 (ii) $\tau^-(b) = \chi^-(b)$;

 (iii) *we have $\pi^i(b) = \Psi^i(b)$ for all $i \in \mathbb{Z}$ and $\gamma = \delta$ on the subset $\{\Psi^i(b) \mid i \in \mathbb{Z}\}$ of $\mathcal{D}$.*

Proof. Firstly we prove $\tau^+(b) \geq \chi^+(b)$ and the equivalence (i) $\Leftrightarrow$ (iii) of (2) at the same time. Let l be any non-negative integer such that $\delta(\pi^i(b)) = \gamma(\Psi^i(b))$ for all $0 \leq i < l$. It suffices to show **(A1)** $\pi^j(b) = \Psi^j(b)$ for all $0 \leq j \leq l$ and **(A2)** $\delta(\pi^l(b)) \geq \gamma(\Psi^l(b))$. The proof of **(A1)** is by induction on j. The case of $j = 0$ is obvious; if $\pi^j(b) = \Psi^j(b)$ $(j < l)$ holds, then the assumption $\delta(\pi^j(b)) = \gamma(\Psi^j(b))$ implies $\pi^{j+1}(b) = \Psi^{j+1}(b)$ by Lemma 4.5 (2); thus we have (**A1**). Thus we have $\pi^l(b) = \Psi^l(b)$ in particular; then by Lemma 4.5 (1) we have **(A2)**.

Secondly we prove $\tau^-(b) \leq \chi^-(b)$. Let l be any natural number such that $\delta(\pi^{-i}(b)) = \gamma(\Psi^{-i}(b))$ for all $1 \leq i < l$. We claim **(B1)** $\pi^{-j}(b) \leq \Psi^{-j}(b)$ for all $0 \leq j < l$ and **(B2)** $\delta(\pi^{-l}(b)) \leq \gamma(\Psi^{-l}(b))$. The proof of **(B1)** is by induction on j. The case of $j = 0$ is obvious; if $\pi^{-(j-1)}(b) \leq \Psi^{-(j-1)}(b)$ $(1 \leq j < l)$ holds, then the assumption $\delta(\pi^{-j}(b)) = \gamma(\Psi^{-j}(b))$ implies that $\pi^{-j}(b) \leq \Psi^{-j}(b)$ for $1 \leq j < l$ by Lemma 4.6 (1); thus we have (**B1**). From $\pi^{-(l-1)}(b) \leq \Psi^{-(l-1)}(b)$, we have **(B2)** $\delta(\pi^{-l}(b)) \leq \gamma(\Psi^{-l}(b))$ by Lemma 4.6 (2).

Since (iii) $\Rightarrow$ (ii) is obvious, it suffices to show (ii) $\Rightarrow$ (i). Obviously (ii) says that for all $i \in \mathbb{Z}_{\geq 1}$ we have $\delta(\pi^{-i}(b)) = \gamma(\Psi^{-i}(b))$. Since both of π and Ψ make cycles, there exists $N \in \mathbb{Z}_{\geq 1}$ such that $\pi^N(b) = b$ and $\Psi^N(b) = b$. For any $j \in \mathbb{Z}_{\geq 0}$, choosing $c \in \mathbb{Z}_{\geq 1}$ with $j - cN < 0$, we have $\delta(\pi^{j-cN}(b)) = \gamma(\Psi^{j-cN}(b))$; hence $\delta(\pi^j(b)) = \gamma(\Psi^j(b))$. Thus we obtain (i). ☐

Corollary 4.8. *Let E be the set $\{b \in \mathcal{D} \mid \tau^+(b) = \chi^+(b)\}$. Then E is π-stable and Ψ-stable and we have $\gamma = \delta$ on E. (Hence $\mathcal{E} := (E, \gamma|_E)$ can be seen as a direct factor of $\mathcal{B}$ and also as a direct factor of $(\mathcal{D}, \gamma)$, see the last sentence of §2.3.)*

Proof. Let b be any element of E. The equality $\gamma(b) = \delta(b)$ is obvious from the equivalence (i) $\Leftrightarrow$ (iii) of Proposition 4.7 (2). This equivalence also implies $\pi(b) \in E$, since b satisfies (iii) if and only if $\pi(b)$ satisfies (iii). Thus we obtain $\pi(E) = E$. Similarly we have $\Psi(E) = E$. ☐

4.3 Slices and strings

The main reference is [10], §2. Also see [7], §4. We recall the definition of slices and strings (cf. [10], §2) in terms of final types.

Definition 4.9. Let $\mathcal{B}_1 = (B_1, \delta_1)$ and $\mathcal{B}_2 = (B_2, \delta_2)$ be final types and set $\pi_1 = \pi_{\delta_1}$ and $\pi_2 = \pi_{\delta_2}$.

(1) A *finite slice* ω is a subset of $B_1 \times B_2$ of the form

$$\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\} \qquad \text{with} \quad |\omega| = \ell$$

for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying

 (a) $\delta_1(s_1) = 1$ and $\delta_2(s_2) = 0$,

(b) $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \le i < \ell$ and

(c) $\delta_1(\pi_1^\ell(s_1)) = 0$ and $\delta_2(\pi_2^\ell(s_2)) = 1$.

We denote by $\Omega_f = \Omega_f(\mathcal{B}_1, \mathcal{B}_2)$ the set of finite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$.

(2) An *infinite slice* ω is a subset of $B_1 \times B_2$ of the form

$$\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \le i \le \ell\} \qquad \text{with} \quad |\omega| = \ell$$

for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying

(a) $s_1 = \pi_1^\ell(s_1)$ and $s_2 = \pi_2^\ell(s_2)$,

(b) $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \le i < \ell$.

We denote by $\Omega_\infty = \Omega_\infty(\mathcal{B}_1, \mathcal{B}_2)$ the set of infinite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Set $\Omega = \Omega(\mathcal{B}_1, \mathcal{B}_2) := \Omega_f \sqcup \Omega_\infty$. An element of Ω is called a *slice*.

Let k be an algebraically closed field of characteristic p.

Definition 4.10. Let $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \le i \le \ell\}$ be a slice with $|\omega| = \ell$. For an element $r \in \omega$, we denote by $\eta(r)$ $(= \eta_\omega(r))$ the integer η with $0 \le \eta < \ell$ satisfying $r = (\pi_1^{\eta+1}(s_1), \pi_2^{\eta+1}(s_2))$.

(1) Let ω be a finite slice. A *string* of ω is the map

$$\psi_{\omega,a} : \quad B_1 \times B_2 \longrightarrow k$$

sending $r \in \omega$ to $a^{p^{\eta(r)}}$ and $r \notin \omega$ to 0 for an element a of k.

(2) Let ω be an infinite slice. A *string* of ω is the map

$$\psi_{\omega,a} : \quad B_1 \times B_2 \longrightarrow \mathbb{F}_{p^{|\omega|}}$$

sending $r \in \omega$ to $a^{p^{\eta(r)}}$ and $r \notin \omega$ to 0 for an element a of $\mathbb{F}_{p^{|\omega|}}$.

Let G_1 and G_2 be BT_1's over k having final types $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. There is a canonical isomorphism as additive groups

$$\Lambda : \quad \prod_{\omega \in \Omega_f} k \times \prod_{\omega' \in \Omega_\infty} \mathbb{F}_{p^{|\omega'|}} \xrightarrow{\ \sim\ } \mathrm{Hom}_k(G_1, G_2).$$

See [10], (2.4). Recall the definition of Λ. Let (ω, a) be a pair of slice ω and $a \in k$ such that $a \in \mathbb{F}_{p^{|\omega|}}$ if ω is infinite. To (ω, a) we associate an element $f_{\omega,a}$ of $\mathrm{Hom}(\mathbb{D}(G_1), \mathbb{D}(G_2))$ as follows. Write $B_* = \{b_1^{(*)} < \cdots < b_{h_*}^{(*)}\}$ for $(* = 1, 2)$ and write $\mathbb{D}(G_*) = \bigoplus_{i=1}^{h_*} kZ_i^{(*)}$ for $(* = 1, 2)$ as in (2.3.3). Then putting $r_{ij} = (b_i^{(1)}, b_j^{(2)}) \in B_1 \times B_2$, we define

$$f_{\omega,a}(Z_i^{(1)}) = \sum_j \psi_{\omega,a}(r_{ij}) Z_j^{(2)}. \qquad (4.3.1)$$

Lemma 4.11. *There exists an injective homomorphism $G_1 \to G_2$ if there exists an injective map $\jmath : B_1 \to B_2$ (as sets) such that for any $b \in B_1$ there exists a finite slice containing $(b, \jmath(b))$.*

Proof. Set $\mathfrak{r}(b) = (b, \jmath(b)) \in B_1 \times B_2$ for $b \in B_1$. Let $\{\omega_1, \ldots, \omega_v\}$ be a minimal set of distinct finite slices such that $\bigcup_i \omega_i$ contains $\mathfrak{r}(B_1)$. For each $b \in B_1$, let $\imath(b)$ be the unique element of $\{1, \ldots, v\}$ with $\mathfrak{r}(b) \in \omega_{\imath(b)}$.

We show that the sum $\sum_i f_{\omega_i,a_i}$ gives an injection $\mathbb{D}(G_1) \to \mathbb{D}(G_2)$ for sufficiently general $(a_i) \in k^v$. Let $\Gamma_{\omega,a}$ denote the $|B_1| \times |B_2|$-matrix $\big(\psi_{\omega,a}(b^{(1)}, b^{(2)})\big)$, where $(b^{(1)}, b^{(2)}) \in B_1 \times B_2$. Note $\Gamma_{\omega,a}$ is the matrix expression of $f_{\omega,a}$, see (4.3.1). Put $\Gamma := \sum_i \Gamma_{\omega_i,a_i}$ and write $\Gamma = (\gamma_{b,b'})$, where $(b, b') \in B_1 \times B_2$. It suffices to show that $\operatorname{rk}\Gamma = |B_1|$ for sufficiently general $(a_i) \in k^v$. From now on we consider $\{a_i\}$ as a set of independent indeterminates. Let $C = (c_{b,b'})_{(b,b') \in B_1 \times B_1}$ be the $|B_1| \times |B_1|$-matrix defined by $c_{b,b'} = \gamma_{b,\jmath(b')}$. It is enough to show that $\det(C) \neq 0$. Consider the terms of

$$\det(C) = \sum_{\zeta \in \mathrm{Aut}(B_1)} (-1)^{\mathrm{sgn}(\zeta)} \prod_{b \in B_1} c_{b,\zeta(b)}. \tag{4.3.2}$$

Note that each entry $c_{b,b'}$ is zero or is of the form $a_i^{p^j}$ for some $i \in \{1, \ldots, v\}$ and for some $j \in \mathbb{Z}_{\geq 0}$, and moreover we have $c_{b,b'} = a_i^{p^e}$ if and only if $(b, \jmath(b')) \in \omega_i$ and $\eta_{\omega_i}(b, \jmath(b')) = e$ (see Definition 4.10 for the definition of η). Hence for every ζ, the term $\prod_{b \in B_1} c_{b,\zeta(b')}$ is zero or is of the form

$$\prod_{i=1}^{v} a_i^{\sum_{e \in S_i} p^e} \tag{4.3.3}$$

for some finite set S_i of non-negative integers. Moreover for any nonzero term, we can recover ζ from the form (4.3.3). Indeed the factor $a_i^{p^e}$ should come from $c_{b,b'}$ for some $b, b' \in B_1$ with $(b, \jmath(b')) \in \omega_i$ and $\eta_{\omega_i}(b, \jmath(b')) = e$, and such a pair (b, b') is unique, since η_{ω_i} is an injective map from ω_i to $\mathbb{Z}_{\geq 0}$ (Definition 4.10); hence ζ is determined by $\zeta(b) = b'$. Thus we need only find a non-zero term of (4.3.2). Writing $\eta_b = \eta_{\omega_{\imath(b)}}$, the term with $\zeta = \mathrm{id}$ is equal to

$$\prod_{b \in B_1} (a_{\imath(b)})^{p^{\eta_b(\mathfrak{r}(b))}} = \prod_{i=1}^{v} a_i^{\sum_{\imath(b)=i} p^{\eta_b(\mathfrak{r}(b))}},$$

which is not zero. $\square$

4.4 Proof of Theorem 1.1

The proof of Theorem 1.1. Let G be a BT_1 of final sequence ν. Set $m = m_\nu$, $n = n_\nu$ and $e = e_\nu$. The aim is to prove that there exists an injective homomorphism $H_{m,n}^{\oplus e}[p] \to G$. Let $\mathcal{B}_1 = (B_1, \delta_1)$ be the final type of $H_{m,n}^{\oplus e}[p]$, and let $\mathcal{B}_2 = (B_2, \delta_2)$ be the final type of G. Put $\mathcal{D} := \mathcal{D}_\nu$ $(\subset B_2)$ and let γ be the partition map defined in Definition 4.4. Recall (4.2.1) that there exists an isomorphism as final types

$$\kappa : \quad \mathcal{B}_1 \xrightarrow{\ \sim\ } (\mathcal{D}, \gamma). \tag{4.4.1}$$

Let $\tau_1^{\pm}, \chi_1^{\pm}$ and $\tau_2^{\pm}, \chi_2^{\pm}$ be the functions defined in (4.2.3) and (4.2.4) for $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Set $E := \{c \in \mathcal{D} \mid \chi_2^{+}(c) = \tau_2^{+}(c)\}$ and write $\mathcal{E} := (E, \gamma|_E)$. By Corollary 4.8, the final type $\mathcal{E}$ can be seen as a direct factor of both of $\mathcal{B}_2$ and $(\mathcal{D}, \gamma)$. Hence removing the direct factors $\kappa^{-1}(\mathcal{E})$ and $\mathcal{E}$ from $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, we show the existence of an injection from the remaining direct factor of $\mathcal{B}_1$ to that of $\mathcal{B}_2$. By Lemma 4.11, it suffices to show that for any $b \in B_1 \setminus \kappa^{-1}(E)$ there exists a finite slice containing $(b, \kappa(b))$. Let $b \in B_1 \setminus \kappa^{-1}(E)$ and put $c := \kappa(b)$. By Proposition 4.7, we have $\chi_2^+(c) < \tau_2^+(c)$ and $\chi_2^-(c) < \tau_2^-(c)$.

Since $\tau_1^+(b) = \chi_2^+(c)$ by (4.4.1), we have $\tau_1^+(b) < \tau_2^+(c)$. Hence there exists $\ell^+ \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{cases} \delta_1(\pi_1^i(b)) = \delta_2(\pi_2^i(c)) & \text{for} \quad 0 \leq i < \ell^+, \\ \delta_1(\pi_1^i(b)) = 0 \quad \text{and} \quad \delta_2(\pi_2^i(c)) = 1 & \text{for} \quad i = \ell^+. \end{cases}$$

Since $\tau_1^-(b) = \chi_2^-(c)$ by (4.4.1), we have $\tau_1^-(b) > \tau_2^-(c)$. Hence there exists $\ell^- \in \mathbb{Z}_{\leq -1}$ such that

$$\begin{cases} \delta_1(\pi_1^i(b)) = \delta_2(\pi_2^i(c)) & \text{for} \quad \ell^- < i < 0, \\ \delta_1(\pi_1^i(b)) = 1 \quad \text{and} \quad \delta_2(\pi_2^i(c)) = 0 & \text{for} \quad i = \ell^-. \end{cases}$$

Thus $\{(\pi_1^i(b), \pi_2^i(c)) \mid \ell^- < i \leq \ell^+\}$ is a finite slice. □

5 Extensions by minimal p-divisible groups

In this section we prove Theorem 1.2 and some corollaries. Let k be an algebraically closed field of characteristic p and set $W = W(k)$ and $A = A_k$. In this section all p-divisible groups and all BT_1's will be over k, and all Dieudonné modules will be over W.

5.1 Proof of Theorem 1.2

It suffices to prove the following, which is a stronger assertion than Theorem 1.2.

Proposition 5.1. *Let $M_1 = \mathbb{D}(H_{c,d} \otimes k)$ with $c, d \geq 0$ and $\gcd(c, d) = 1$. Let Q be a DM_1 and ϕ a surjective A-homomorphism $Q \to M_1/pM_1$. Set $P = \operatorname{Ker} \phi$. (Note P is also a DM_1 by Lemma 2.2). Then for any free Dieudonné module M_2 such that $M_2/pM_2 \simeq P$, there exist a free Dieudonné module M, a surjection $f : M \to M_1$ and an isomorphism $g : M/pM \simeq Q$ commuting*

$$\begin{array}{ccc} M & \xrightarrow{f} & M_1 \\ \downarrow & & \downarrow \\ M/pM & \xrightarrow{\phi \circ g} & M_1/pM_1 \end{array}$$

such that $\operatorname{Ker} f \simeq M_2$.

Proof. If c or d is zero, then the homomorphism $\phi : Q \to M_1/pM_1$ has a splitting; hence the proposition holds obviously. From now on we assume $c, d > 0$.

Let $u = \min\{c, d\}$. Recall Lemma 3.3 that M_1 is generated over A by $X_1, \cdots, X_u$ and all relations are generated over A by $\mathcal{F}^{\alpha_i} X_i - \mathcal{V}^{\beta_{i+1}} X_{i+1} = 0$.

Let $\sum_{l=1}^t (m_l, n_l)$ be the Newton polygon of M_2. Then

$$M_2 \otimes_W \operatorname{frac}(W) \xrightarrow{\sim} \bigoplus_{l=1}^t \mathbb{D}(H_{m_l,n_l}) \otimes_W \operatorname{frac}(W). \tag{5.1.1}$$

Let $e_l \in M_2 \otimes_W \mathrm{frac}(W)$ be the highest element of $\mathbb{D}(H_{m_l,n_l})$. Let ϑ_l be the endomorphism of H_{m_l,n_l} defined just after Definition 3.1. We define a commutative discrete valuation ring

$$R_l := W[\theta_l]/(\theta_l^{m_l+n_l} - p)$$

and set $L_l = \mathrm{frac}(W)[\theta_l]/(\theta_l^{m_l+n_l} - p) = \mathrm{frac}(R_l)$. We extend the action of the Frobenius σ on W to that on L_l by the rule $\theta_l^\sigma = \theta_l$. Note the W-homomorphism

$$R_l \longrightarrow \mathbb{D}(H_{m_l,n_l} \otimes k) \tag{5.1.2}$$

defined by sending $f(\theta_l)$ to $f(\vartheta_l)e_l$ is isomorphic.

Let $Y_1, \cdots, Y_w$ be a W-basis of M_2. Since there is an exact sequence

$$0 \longrightarrow M_2/pM_2 \longrightarrow Q \longrightarrow M_1/pM_1 \longrightarrow 0, \tag{5.1.3}$$

it follows that Q is generated over $k[\mathcal{F},\mathcal{V}]$ by $\overline{Y}_1, \cdots, \overline{Y}_w, \mathcal{F}^r\overline{Z}_i, \mathcal{V}^s\overline{Z}_i$ ($0 \le r \le \alpha_i$ and $0 \le s < \beta_i$) and all relations are generated over $k[\mathcal{F},\mathcal{V}]$ by relations only in $\overline{Y}_1, \cdots, \overline{Y}_w$ and relations of the forms

$$\mathcal{F}^{\alpha_i}\overline{Z}_i - \mathcal{V}^{\beta_{i+1}}\overline{Z}_{i+1} = \sum_j c_{ij}\overline{Y}_j \tag{5.1.4}$$

with $c_{ij} \in k$.

We will define M to be an A-submodule of $(M_1 \oplus M_2) \otimes_W \mathrm{frac}(W)$ generated by M_2 and

$$\mathcal{F}^r Z_i \quad (0 \le r \le \alpha_i) \qquad \text{and} \qquad \mathcal{V}^s Z_i \quad (0 \le s < \beta_i)$$

for $i = 1, 2, \cdots, n$ where Z_i is of the form:

$$Z_i = X_i + \sum_{l=1}^{t} a_{il}e_l$$

for some $a_{il} \in L_l$, which will be chosen later so that M has the required properties.

Let $\tilde{c}_{ij} \in W$ be a lift of c_{ij} and define $b_{il} \in L_l$ by

$$\sum_l b_{il}e_l = \sum_j \tilde{c}_{ij}Y_j.$$

It suffices to show that there exists a solution $\{a_{il}\}$ ($1 \le i \le u,\ 1 \le l \le t$) satisfying

$$\mathcal{F}^{\alpha_i}Z_i - \mathcal{V}^{\beta_{i+1}}Z_{i+1} = \sum_l b_{il}e_l. \tag{5.1.5}$$

Comparing the coefficients of e_l of the both sides of (5.1.5), we obtain

$$a_{il}^{\sigma^{\alpha_i}}\theta_l^{n_l\alpha_i} - a_{i+1,l}^{\sigma^{-\beta_{i+1}}}\theta_l^{m_l\beta_{i+1}} = b_{il} \tag{5.1.6}$$

for $i \in \mathbb{Z}/u\mathbb{Z}$. Since l is the same in each equation, it suffices to solve the simultaneous equations for each l. Writing a_i, b_i, n, m, θ and L for a_{il}, b_{il}, n_l, m_l, θ_l and L_l respectively, we have

$$a_1^{\sigma^{\sum_{i=1}^u(\alpha_i+\beta_i)}}\theta^{\sum_{i=1}^u(n\alpha_i-m\beta_i)} - a_1 = \sum_{i=1}^{u} b_i^{\sigma^{\beta_1+\sum_{i<j\le u}(\alpha_j+\beta_j)}}\theta^{-m\beta_1+\sum_{i<j\le u}(n\alpha_j-m\beta_j)}. \tag{5.1.7}$$

It suffices to show that this has a solution $a_1 \in L$; then we get a required solution $\{a_i\}_{i=1}^u$ from (5.1.6).

Put $z := a_1$ and $\varrho := \sigma^{\sum_{i=1}^u(\alpha_i+\beta_i)}$. Note $\varrho \neq 1$ by $\alpha_i, \beta_i > 0$. We also put $\epsilon := \sum_{i=1}^u(n\alpha_i - m\beta_i)$ and $v :=$ RHS of (5.1.7). Then (5.1.7) is written as

$$z^\varrho \theta^\epsilon - z = v. \tag{5.1.8}$$

If $\epsilon > 0$, we have a solution $z = \sum_{\ell=0}^\infty \theta^{\ell\epsilon}(-v)^{\varrho^\ell}$. Also if $\epsilon < 0$, we have a solution $z = \sum_{\ell=1}^\infty \theta^{-\ell\epsilon} v^{\varrho^{-\ell}}$. Finally we consider the case $\epsilon = 0$. Write $z = \sum_{i=0}^{m+n-1} z_i\theta^i$ and $v = \sum_{i=0}^{m+n-1} v_i\theta^i$ with $z_i, v_i \in \mathrm{frac}(W)$. It suffices to solve $z_i^\varrho - z_i = v_i$ for each $0 \le i < m+n$. There exist elements y_j of W for all integers $j \ge \mathrm{ord}_p(v_i)$ such that $z_i = \sum_{j=\mathrm{ord}_p(v_i)}^\infty p^j y_j$ is a solution. Indeed, putting $z_{ij} := \sum_{j'<j} p^{j'} y_{j'}$, we can find $y_{j'}$ successively so that $z_{ij}^\varrho - z_{ij} \equiv v_i \pmod{p^j W}$. Let $j \ge \mathrm{ord}_p(v_i)$ and suppose that we have already got such $y_{j'}$ for $j' < j$. Since $\varrho \neq 1$, there exists a solution $\overline{y}_j \in k$ of the Artin-Schreier equation $\overline{y}_j^\varrho - \overline{y}_j = (p^{-j}(v_i - z_{ij}^\varrho + z_{ij}) \mod pW)$. Let y_j be a lift of $\overline{y}_j$. Then clearly $z_i := \sum_{j=\mathrm{ord}_p(v_i)}^\infty p^j y_j$ is a solution of $z_i^\varrho - z_i = v_i$. □

We obtain the "dual" of Proposition 5.1:

Proposition 5.2. *Let $M_1 = \mathbb{D}(H_{c,d})$ with $c, d \ge 0$ and $\gcd(c,d) = 1$. Let Q be a DM_1 and ϕ an injective A-homomorphism $M_1/pM_1 \to Q$. Set $P = \mathrm{Coker}\,\phi$. (Note P is also a DM_1 by Lemma 2.2). Then for any free Dieudonné module M_2 such that $M_2/pM_2 \simeq P$, there exist a free Dieudonné module M, an injective A-homomorphism $f : M_1 \to M$ and an isomorphism $g : Q \simeq M/pM$ commuting*

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M_1/pM_1 & \xrightarrow{g\circ\phi} & M/pM \end{array}$$

such that $\mathrm{Coker}\, f \simeq M_2$.

Proof. Let N be a free Dieudonné module and let N_1 be a DM_1. Their duals are defined by $\check{N} := \mathrm{Hom}_W(N, W)$ and $\check{N}_1 := \mathrm{Hom}_k(N_1, k)$ with $\mathcal{F}$ and $\mathcal{V}$-operations defined by $(\mathcal{F}\varphi)(x) = \varphi(\mathcal{V}x)^\sigma$ and $(\mathcal{V}\varphi)(x) = \varphi(\mathcal{F}x)^{\sigma^{-1}}$, where $(\varphi, x) \in \check{N} \times N$ and $(\varphi, x) \in \check{N}_1 \times N_1$ respectively. The dual of N/pN is canonically isomorphic to $\check{N}/p\check{N}$.

We can apply $(\check{M}_1, \check{Q}, \check{\phi}, \check{P}, \check{M}_2)$ to (M_1, Q, ϕ, P, M_2) in Proposition 5.1. Then there exist a free Dieudonné module M', a surjection $f' : M' \to \check{M}_1$ and an isomorphism $g' : M'/pM' \simeq \check{Q}$ commuting

$$\begin{array}{ccc} M' & \xrightarrow{f'} & \check{M}_1 \\ \downarrow & & \downarrow \\ M'/pM' & \xrightarrow{\check{\phi}\circ g'} & \check{M}_1/p\check{M}_1 \end{array}$$

such that $\mathrm{Ker}\, f' \simeq \check{M}_2$. Then $(M, f, g) := (\check{M}', \check{f}', \check{g}')$ satisfies all required conditions. □

5.2 Proof of Corollary 1.3

Let ν be a final sequence, and set $m = m_\nu$, $n = n_\nu$ and $e = e_\nu$. Recall $\rho_\nu = m/(m+n)$.

Corollary 5.3. *There exists a p-divisible group $\mathcal{G}$ such that*

(i) $\mathrm{FS}(\mathcal{G}[p]) = \nu$ *and*

(ii) $\rho_1(\mathcal{G}) = \cdots = \rho_e(\mathcal{G}) = \rho_\nu$. *(See* (2.2.3) *for the definition of $\rho_i(\mathcal{G})$.)*

Proof. This follows immediately from Theorem 1.1, Proposition 5.2 and Proposition 4.2. □

Then Corollary 1.3 follows from Proposition 4.2 and Corollary 5.3.

Corollary 5.4. $\rho_\nu = \max\{c/(c+d) \mid \nu_{c,d}$ *is embeddable into* $\nu\}$.

Proof. Theorem 1.1 says that $\nu_{m,n}$ is embeddable into ν; hence LHS $\leq$ RHS. If $\nu_{c,d}$ is embeddable into ν, then it follows from Theorem 1.2 that there exists a p-divisible group $\mathcal{G}$ with $\mathrm{FS}(\mathcal{G}[p]) = \nu$ containing slope $c/(c+d)$. Note $c/(c+d)$ is less than or equal to the last Newton slope $\rho_1(\mathcal{G})$ of $\mathcal{G}$, and we have $\rho_1(\mathcal{G}) \leq \rho_\nu$ by Corollary 1.3. Hence we have LHS $\geq$ RHS. □

5.3 Proof of Corollary 1.4

The proof of Corollary 1.4. The "only if"-part follows immediately from Theorem 1.1. Let us give two proofs of the "if"-part. Let G be a minimal and indecomposable BT_1, i.e., $G \simeq H_{c,d}[p]$ for non-negative integers c, d with $\gcd(c, d) = 1$. Let ν be the final sequence of G. Assume G were not BT_1-simple. From this assumption, G can be written as a successive extension of $\{H_{m_i,n_i}[p]\}_{i=1}^t$ for some $t \geq 2$ by Theorem 1.1 and Lemma 2.2. Note $c = \sum_{i=1}^t m_i$ and $d = \sum_{i=1}^t n_i$. Then by Theorem 1.2, there exists a p-divisible group $\mathcal{G}$ such that $\mathcal{G}[p] \simeq G$ and $\mathcal{G}$ is a successive extension of $\{H_{m_i,n_i}\}_{i=1}^t$. Note

$$\mathrm{NP}(\mathcal{G}) = \sum_{i=1}^{t} (m_i, n_i). \tag{5.3.1}$$

First proof: The last Newton slope $m_t/(m_t + n_t)$ of $\mathcal{G}$ is greater than or equal to $c/(c + d)$ by (5.3.1). Corollary 1.3 says the last Newton slope is at most $c/(c+d)$. Hence we get $m_t/(m_t+n_t) = c/(c + d)$. From $\gcd(c, d) = 1$ we have $(c, d) = (m_t, n_t)$. This contradicts with $t \geq 2$.
Second proof: We use Oort's result [9]:

> *Let X be a p-divisible group over an algebraically closed field k. If $X[p] \simeq H(\xi)[p] \otimes k$, then $X \simeq H(\xi) \otimes k$ over k.*

Then $\mathcal{G} \simeq H_{c,d}$ has to hold, since $\mathcal{G}[p] \simeq H_{c,d}[p]$. However this contradicts with (5.3.1) and $t \geq 2$. □

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