GENERIC NEWTON POLYGONS OF EKEDAHL-OORT STRATA: OORT'S CONJECTURE

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Dedicated to Professor Toshiyuki Katsura on his 60th birthday

Abstract. We study the moduli space of principally polarized abelian varieties in positive characteristic. In this paper we determine the Newton polygon of any generic point of each Ekedahl-Oort stratum, by proving Oort's conjecture on intersections of Newton polygon strata and Ekedahl-Oort strata. This result tells us a combinatorial algorithm determining the optimal upper bound of the Newton polygons of principally polarized abelian varieties with a given isomorphism type of p-kernel.

1. Introduction

We fix once for all a rational prime p. For an abelian variety A over an algebraically closed field of characteristic p, we have two objects: the p-divisible group $A[p^\infty]$ and the p-kernel $A[p]$, a truncated Barsotti-Tate group of level one (BT_1). By the Dieudonné-Manin classification, the isogeny classes of p-divisible groups are classified by Newton polygons (cf. §2.2). On the other hand, the isomorphism classes of polarized BT_1's are classified by final elements of the Weyl group $\mathbb{W}_g$ of the symplectic group Sp_{2g} (cf. §4.2). For a BT_1 G, we write $G \simeq w$ if the isomorphism type of G is w. The following question is still open in general:

> *For a final element w of $\mathbb{W}_g$, which Newton polygons can occur as the Newton polygons $\mathcal{N}(A)$ of principally polarized abelian varieties (A, η) with $A[p] \simeq w$?*

A purpose of this paper is to give a combinatorial algorithm determining the optimal upper bound $b(w)$ of such Newton polygons. The precise definition of $b(w)$ is as follows: any (A, η) with $A[p] \simeq w$ satisfies $\mathcal{N}(A) \prec b(w)$ and there exists (A', η') satisfying $A'[p] \simeq w$ and $\mathcal{N}(A') = b(w)$. We shall explain below the non-trivial fact that $b(w)$ exists.

In order to accomplish the purpose above, we investigate some stratifications and foliations on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g in characteristic p. For a symmetric Newton polygon ξ, we write $\mathcal{W}^0_\xi$ for the open Newton polygon stratum (cf. §2.2). For a final element w of $\mathbb{W}_g$, let $\mathcal{S}_w$ be the Ekedahl-Oort stratum:

$$\mathcal{S}_w = \{(A, \eta) \in \mathcal{A}_g \mid A[p] \simeq w\}.$$

In §4.3 we will give a brief review of some known facts on the Ekedahl-Oort stratification. Among those, Oort showed that $\mathcal{S}_w \neq \emptyset$ for every final element w of $\mathbb{W}$

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and Ekedahl and van der Geer proved that $\mathcal{S}_w$ is irreducible if $\mathcal{S}_w$ is not contained in the supersingular locus. The *generic Newton polygon* $\xi(w)$ of $\mathcal{S}_w$ is defined to be the Newton polygon of the generic point of $\mathcal{S}_w$ if $\mathcal{S}_w$ is not contained in the supersingular locus and otherwise the supersingular Newton polygon. Since the Newton polygon goes down or stays w.r.t. $\prec$ under any specialization (Grothendieck-Katz [15], Th. 2.3.1 on p. 143), we see that $\xi(w)$ fulfills the conditions defining $b(w)$; thus $b(w)$ exists and $\xi(w) = b(w)$.

Let $\mathcal{Z}_\xi$ be the central stream in $\mathcal{A}_g$ of the Newton polygon ξ (cf. §5.3) and let $\overline{\mathcal{S}_w}$ denote the Zariski closure of $\mathcal{S}_w$ in $\mathcal{A}_g$. We shall show

Main theorem. *For any final element w of $\mathbb{W}_g$, we have $\mathcal{Z}_{\xi(w)} \subset \overline{\mathcal{S}_w}$.*

The main theorem is closely related to [24], (6.9):

Oort's conjecture. *If $\mathcal{W}^0_\xi \cap \mathcal{S}_w \neq \emptyset$, then $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$.*

Indeed in [11], Cor. 3.7, it was proved that the main theorem and the conjecture are equivalent. Thus we obtain

Corollary I. *Oort's conjecture is true.*

Here is another corollary. Let ξ be any symmetric Newton polygon with $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$. Since the Newton polygon of every point of $\mathcal{Z}_\xi$ is ξ and the generic Newton polygon of $\overline{\mathcal{S}_w}$ is $\xi(w)$, we have $\xi \prec \xi(w)$ by Grothendieck-Katz. Hence the main theorem implies

Corollary II. *$\xi(w)$ is the biggest element of the set $\{\ \xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\ \}$ with respect to $\prec$.*

This gives a purely combinatorial algorithm determining $\xi(w)$. Indeed we have $\mathcal{Z}_\xi = \mathcal{S}_{w_\xi}$ for a certain final element $w_\xi \in \mathbb{W}$ (cf. §5.3), and there is an algorithm determining w_ξ for a concretely given ξ (see [11], Cor. 4.27); by using Wedhorn's result in [28] (see Th. 4.3.2 below for a copy) and Rem. 4.3.3 we can check whether $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$ for a concretely given ξ and w; thus it is possible to describe the set $\{\xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\}$ for a given w; finally find the biggest element in the set, which exists and is equal to $\xi(w)$.

See [9] for a more effective algorithm determining the first slope of $\xi(w)$. We see a beautiful similarity between Cor. II and the result [7], Th. 5.4.11 of Goren and Oort in the case of Hilbert modular varieties over inert primes.

Let us explain the structure of this paper. The first five sections consist of preliminaries, where we recall some fundamental facts on p-divisible groups, F-zips, stratifications and foliations on $\mathcal{A}_g$ and we also prove some auxiliary results used later on. The heart of this paper is Section 6, where we show that, to prove the main theorem, it suffices to construct a certain family of p-divisible groups with constant Newton polygon and with constant p-kernel type (Th. 6.1.1), and then give the reader our idea on how to construct such a family. The remaining sections are devoted to realizing the construction. The key propositions for the construction are Prop. 7.6.1 and 8.3.1. In Prop. 7.6.1 we construct a non-trivial self-dual complex of F-zips and in Prop. 8.3.1 we lift such a self-dual complex of F-zips to a self-dual complex of displays.

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Notations.

- $\mathbb{N} = \mathbb{Z}_{>0}$ the set of natural numbers.
- For $m, n \in \mathbb{Z}_{\geq 0}$, we denote by $\gcd(m, n)$ the greatest common divisor, where for convenience we set $\gcd(m, 0) = \gcd(0, m) = m$ for $\forall m \in \mathbb{Z}_{\geq 0}$. We say that $m, n \in \mathbb{Z}_{\geq 0}$ are coprime if $\gcd(m, n) = 1$.
- For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the biggest integer $\leq x$ and $\lceil x \rceil$ the smallest integer $\geq x$.
- For an integral domain R, we denote by $\mathrm{frac}(R)$ the fractional field of R.
- $W(R)$ the ring of Witt vectors with coordinates in R.
- $\mathbb{W}_g$ the Weyl group of the symplectic group Sp_{2g}.
- ${}^J\mathbb{W}$ the set of final elements in $\mathbb{W}_g$.
- $Z^\vee$ the dual of an F-zip Z.
- $G^\vee$ the Cartier dual of a commutative finite group scheme G.
- X^t the Serre dual of a p-divisible group X.
- M^t the dual Dieudonné module (display) of a Dieudonné module (display) M.
- A^t the dual abelian variety of an abelian variety A.
- $\mathcal{A}_g$ the moduli of principally polarized abelian varieties in characteristic p.
- $\mathcal{W}_\xi$ the Newton polygon stratum for a symmetric Newton polygon ξ.
- $\mathcal{C}_x, \mathcal{C}_{(X,\imath)}$ the central leaf for $x \in \mathcal{A}_g$ or a principally quasi-polarized p-divisible group $(X, \imath)$.
- $\mathcal{I}_x$ the isogeny leaf for $x \in \mathcal{A}_g$.
- $\mathcal{Z}_\xi$ the central stream for a symmetric Newton polygon ξ.
- $\mathcal{S}_w$ the Ekedahl-Oort stratum for $w \in {}^J\mathbb{W}$.
- $H(\xi)$ the minimal p-divisible group of ξ.
- w_ξ the element of ${}^J\mathbb{W}$ corresponding to the p-kernel of $H(\xi)$.
- $\xi(w)$ the generic Newton polygon of $\mathcal{S}_w$ for $w \in {}^J\mathbb{W}$.

2. p-DIVISIBLE GROUPS

We start with reviewing the display theory (Zink [29]) on the classification of p-divisible groups. Also we recall the definition of Newton polygon stratification.

For a commutative ring R, let $W(R)$ denote the ring of Witt vectors with coordinates in R. Let $\sigma : x \mapsto {}^\sigma x$ be the Frobenius on $W(R)$ and let $\tau : x \mapsto {}^\tau x$ be the Verschiebung on $W(R)$. Put $I_R = {}^\tau W(R)$ and $I_{R,n} = {}^{\tau^n} W(R)$ for $n \in \mathbb{N}$, which are ideals of $W(R)$.

2.1. **Displays.** First we briefly review the Dieudonné theory. Let K be a perfect field of characteristic p. Let E_K be the p-adic completion of the associative ring

$$W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}x - {}^\sigma x\mathcal{F}, \mathcal{V}{}^\sigma x - x\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \forall x \in W(K)). \tag{2.1}$$

A Dieudonné module over $W(K)$ is a left E_K-module which is finitely generated as a $W(K)$-module. The covariant Dieudonné theory says that there is a canonical categorical equivalence $\mathbb{D}$ from the category of p-divisible groups (resp. p-torsion finite commutative group schemes) over K to the category of Dieudonné modules over $W(K)$ which are free as $W(K)$-modules (resp. of finite length). Note that $\mathbb{D}$ satisfies $\mathbb{D}(G) = \mathrm{M}(G^\vee)$ for a finite commutative group scheme G, where $G^\vee$ is the Cartier dual of G and M is the contravariant Dieudonné functor (cf. [4], Chap. III). We write F and V for "Frobenius" and "Verschiebung" on commutative group schemes. The covariant Dieudonné functor $\mathbb{D}$ satisfies $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$.

Zink [29] introduced the notion of display and classified formal p-divisible groups over very wide range of rings, generalizing the Dieudonné theory. For a $W(R)$-module P, we write $P^\sigma = W(R) \otimes_{\sigma, W(R)} P$.

Definition 2.1.1. A *display* over R is a quadruple $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$, where P is a finitely generated projective $W(R)$-module, $Q \subset P$ is a submodule and $\mathcal{F}$ and $\mathcal{V}^{-1}$ are $W(R)$-linear maps $\mathcal{F} : P^\sigma \to P$ and $\mathcal{V}^{-1} : Q^\sigma \to P$ such that

1. $I_R P \subset Q \subset P$ and there exists a decomposition $P = L \oplus T$ as $W(R)$-modules such that $Q = L \oplus I_R T$;
2. $\mathcal{V}^{-1} : Q^\sigma \to P$ is an epimorphism;
3. For $x \in P$ and $w \in W(R)$ we have $\mathcal{V}^{-1}(1 \otimes {}^\tau wx) = w\mathcal{F}(1 \otimes x)$;
4. $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ satisfies the $\mathcal{V}$-nilpotence condition ([29], before Def. 11).

Zink showed [29], Th. 9:

Theorem 2.1.2. *Assume R is an excellent local ring or a ring of finite type over a field k of characteristic p. Then there is a canonical categorical equivalence from the category of displays over R to the category of formal p-divisible groups over R.*

Remark 2.1.3. Let X be a formal p-divisible group over a perfect field K. The display of X is given by the quadruple $(M, \mathcal{V}M, \mathcal{F}, \mathcal{V}^{-1})$, where M is the Dieudonné module of X and the others are naturally defined by the $\mathcal{F}, \mathcal{V}$-operations on M.

2.2. **The NP-stratification.** A pair (m, n) of coprime non-negative integers is called a *segment*. For a series of segments (m_i, n_i) $(i = 1, \ldots, \mathfrak{t})$ satisfying $\lambda_1 \leq \cdots \leq \lambda_\mathfrak{t}$ with $\lambda_i = m_i/(m_i + n_i)$, putting $P_j := (\sum_{i=1}^j (m_i + n_i), \sum_{i=1}^j m_i) \in \mathbb{R}^2$ for $0 \leq j \leq \mathfrak{t}$, we denote by $\sum_i (m_i, n_i)$ the line graph in $\mathbb{R}^2$ passing through $P_0, \ldots, P_\mathfrak{t}$ in this order. We call such a line graph a *Newton polygon*. $\lambda_\mathfrak{t}$ is called the *last Newton slope*. We say, for two Newton polygons ξ, ξ' with the same end point, that $\xi' \prec \xi$ if no point of ξ is below ξ'. A Newton polygon $\sum_i (m_i, n_i)$ is said to be *symmetric* if $\lambda_i + \lambda_{\mathfrak{t}+1-i} = 1$ for all $i = 1, \ldots, \mathfrak{t}$. The symmetric Newton polygon $\sum_i (1, 1)$ is called *supersingular*.

For a segment (m, n), we define a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(G_{m,n}) = E_{\mathbb{F}_p}/E_{\mathbb{F}_p}(\mathcal{F}^m - \mathcal{V}^n). \tag{2.2}$$

By the Dieudonné-Manin classification [18], for any p-divisible group X over a field K of characteristic p, there is an isogeny over an algebraically closed field Ω containing K from X to $\bigoplus_{i=1}^{\mathfrak{t}} G_{m_i, n_i}$ for some finite set $\{(m_i, n_i)\}$ of segments. Thus we get a Newton polygon $\sum_i (m_i, n_i)$, which is denoted by $\mathcal{N}(X)$. For an abelian variety A, we have its Newton polygon $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$. Note that $\mathcal{N}(A)$ is symmetric.

For a symmetric Newton polygon ξ of height $2g$, we define its *NP-stratum* by

$$\mathcal{W}_\xi = \{(A, \eta) \in \mathcal{A}_g \,|\, \mathcal{N}(A) \prec \xi\}.$$

Grothendieck and Katz ([15], Th. 2.3.1 on p. 143) proved that $\mathcal{W}_\xi$ is closed in $\mathcal{A}_g$; we consider this is a closed subscheme by giving it the induced reduced scheme structure. We also define the *open NP-stratum* by

$$\mathcal{W}^0_\xi = \{(A,\eta) \in \mathcal{A}_g \,|\, \mathcal{N}(A) = \xi\};$$

similarly we regard $\mathcal{W}^0_\xi$ as a locally closed subscheme of $\mathcal{A}_g$.

3. The first de Rham cohomology

Let S be a scheme of characteristic p. Let $f : A \to S$ be an abelian scheme over S. Let F_S be the absolute Frobenius and let $f^{(p)} : A^{(p)} \to S$ denote $F_S \times f : S \times_{F_S,S} A \to S$. Let $F : A \to A^{(p)}$ be the relative Frobenius. We consider the first de Rham cohomology sheaf $N = H^1_{\mathrm{dR}}(A/S)$, which is a locally free $\mathcal{O}_S$-module. Recall N is equipped with two canonical subsheaves $C := f_*\Omega^1_{A/S}$ and $D := R^1 f^{(p)}_*(\mathcal{H}^0(F_*\Omega^\bullet_{A/S}))$. The Cartier isomorphism induces canonical isomorphisms $\varphi : (N/C)^{(p)} \to D$ and $\dot\varphi : C^{(p)} \to N/D$. If A has a principal polarization η, it induces an alternating perfect pairing $\langle\ ,\ \rangle$ on N. Thus from (A,η) we have a polarized F-zip

$$\mathrm{Fz}(A,\eta) := (N, C, D, \varphi, \dot\varphi, \langle\ ,\ \rangle). \tag{3.1}$$

We start with reviewing the abstract definition of (polarized) F-zips for the reader's convenience. In this paper if we simply say (polarized) F-zip, it means (symplectic) F-zip of type with support contained in $\{0,1\}$ in the terminology of [21] and [27].

3.1. ***F*-zips.** For an $\mathcal{O}_S$-module $\mathcal{M}$ we write $\mathcal{M}^{(p)} = F_S^*\mathcal{M}$.

Definition 3.1.1. An *F-zip* over S is a quintuple $Z = (N, C, D, \varphi, \dot\varphi)$ consisting of locally free $\mathcal{O}_S$-module N and $\mathcal{O}_S$-submodules C, D of N which are locally direct summands of N, and $\mathcal{O}_S$-linear isomorphisms

$$\varphi : (N/C)^{(p)} \longrightarrow D, \qquad \dot\varphi : C^{(p)} \longrightarrow N/D.$$

If S is connected, we define the *height* of Z to be the rank of N and the *type* of Z to be a map from $\{0,1\}$ to $\mathbb{Z}_{\geq 0}$ sending 0 to $\mathrm{rk}\, D$ and 1 to $\mathrm{rk}\, C$; we will simply write the type as $(\mathrm{rk}\, D, \mathrm{rk}\, C)$.

Definition 3.1.2. Let $Z_1 = (N_1, C_1, D_1, \varphi_1, \dot\varphi_1)$ and $Z_2 = (N_2, C_2, D_2, \varphi_2, \dot\varphi_2)$ be two F-zips over S. The set $Hom_S(Z_1, Z_2)$ of homomorphisms as F-zips consists of elements μ of $Hom_{\mathcal{O}_S}(N_1, N_2)$ such that

(1) $\mu(C_1) \subset C_2$ and $\mu(D_1) \subset D_2$,
(2) $\mu \circ \varphi_1 = \varphi_2 \circ \mu^{(p)}$ and $\mu \circ \dot\varphi_1 = \dot\varphi_2 \circ \mu^{(p)}$.

3.2. **Polarized *F*-zips.** For an $\mathcal{O}_S$-module N, we write $N^\vee = \mathcal{H}om_{\mathcal{O}_S}(N, \mathcal{O}_S)$. A pairing $\langle\ ,\ \rangle : N \otimes_{\mathcal{O}_S} N \to \mathcal{O}_S$ canonically induces a pairing on $N^{(p)}$:

$$\langle\ ,\ \rangle^{(p)} :\quad N^{(p)} \otimes_{\mathcal{O}_S} N^{(p)} \longrightarrow \mathcal{O}_S^{(p)} \simeq \mathcal{O}_S,$$

where the last isomorphism is defined by gluing the canonical isomorphisms $R \otimes_{\sigma,R} R \simeq R$ over affine open subschemes $\mathrm{Spec}(R) \subset S$.

Definition 3.2.1. Let $Z = (N, C, D, \varphi, \dot\varphi)$ be an F-zip. A *polarization* on Z is a perfect alternating pairing on N:

$$\langle\ ,\ \rangle :\quad N \otimes_{\mathcal{O}_S} N \longrightarrow \mathcal{O}_S$$

("alternating" means $\langle z, z\rangle = 0$ for all $z \in N$) such that

(1) C and D are totally isotropic, and
(2) $\langle \dot\varphi y, \varphi x\rangle = \langle y, x\rangle^{(p)}$ for $x \in N^{(p)}$ and $y \in C^{(p)}$. (The LHS makes sense by (1). See [21], (5.2) and [27], (2.3) for an equivalent condition.)

We call such a pair $(Z, \langle\ ,\ \rangle)$ a *polarized F-zip*.

3.3. **The dual of an F-zip.** Let Z be an F-zip $(N, C, D, \varphi, \dot\varphi)$. We define the *dual F-zip* $Z^\vee$ of Z by

$$(N^\vee, (N/C)^\vee, (N/D)^\vee, (\dot\varphi^\vee)^{-1}, (\varphi^\vee)^{-1}).$$

Clearly a homomorphism $f : Z_1 \to Z_2$ induces a homomorphism $f^\vee : Z_2^\vee \to Z_1^\vee$ canonically. Note that a polarization $\langle\ ,\ \rangle$ on Z gives an isomorphism $Z \to Z^\vee$ of F-zips.

3.4. **Truncated Barsotti-Tate groups of level one (BT_1).**

Definition 3.4.1 ([13])**.** Let S be an $\mathbb{F}_p$-scheme. A finite locally free commutative group scheme G over S is said to be a BT_1 if it is annihilated by p and $\mathrm{Im}(V : G^{(p)} \to G) = \mathrm{Ker}(F : G \to G^{(p)})$.

Note that the p-kernel of a p-divisible group is a BT_1. Let K be a perfect field. For a BT_1 G over K, putting $N = \mathbb{D}(G)$ we have an F-zip

$$\mathrm{fz}(G) := (N, \mathcal{V}N, \mathcal{F}N, \mathcal{F}, \mathcal{V}^{-1}). \tag{3.2}$$

Definition 3.4.2. Assume K is perfect. Let G be a BT_1 over K. A *symmetry* of G is an isomorphism from G to its Cartier dual $G^\vee$. A symmetry $\imath$ is called a *polarization* if the bilinear form $\langle\ ,\ \rangle : \mathbb{D}(G) \otimes_K \mathbb{D}(G) \to K$ induced by $\imath$ is alternating: $\langle x, x\rangle = 0$ for all $x \in \mathbb{D}(G)$. A *polarized* BT_1 is a pair $(G, \imath)$ consisting of a BT_1 G and a polarization $\imath$.

For a polarized BT_1 $(G, \imath)$ over K we have a polarized F-zip

$$\mathrm{fz}(G, \imath) := (N, \mathcal{V}N, \mathcal{F}N, \mathcal{F}, \mathcal{V}^{-1}, \langle\ ,\ \rangle), \tag{3.3}$$

where $N = \mathbb{D}(G)$ and $\langle\ ,\ \rangle$ is the polarization induced by $\imath$.

Remark 3.4.3. Over a perfect field, fz makes a categorical equivalence from the category of (polarized) BT_1's to that of (polarized) F-zips. Moreover if $Z = \mathrm{fz}(G)$, then $Z^\vee = \mathrm{fz}(G^\vee)$, where $G^\vee$ is the Cartier dual of G.

3.5. **Displays modulo I_R.** In this subsection we show that the reduction modulo I_R of a display over R defines an F-zip over R.

Let $M = (P, Q, \mathcal{F}, \mathcal{V}^{-1})$ be a display over R. Put $N = P/I_RP$ and $C = Q/I_RP$.

Lemma 3.5.1. *$\mathcal{F}$ induces an injective homomorphism $(N/C)^{(p)} \to N$.*

Proof. Let $P = L \oplus T$ be a normal decomposition. Then

$$C = Q/I_RP = (L \oplus I_RT)/(I_RL \oplus I_RT) = L/I_RL$$

and $N/C = T/I_RT$. Recall that the $W(R)$-linear homomorphism

$$\mathcal{V}^{-1} \oplus \mathcal{F} : \quad L^\sigma \oplus T^\sigma \longrightarrow P \tag{3.4}$$

is an isomorphism ([29], Lem. 9). Note that the map $W(R) \otimes_{\sigma, W(R)} T \to R \otimes_{\sigma, R} (T/I_RT)$ induces a canonical isomorphism from (T^σ/I_RT^σ) to $(T/I_RT)^{(p)} = (N/C)^{(p)}$. Hence we have the injection $\mathcal{F} : (N/C)^{(p)} \to N$. $\square$

We define a submodule D of N to be the image of the injection obtained in Lem. 3.5.1, namely D is the $\mathcal{F}$-image of T^σ in N. Note that D is independent of the choice of the normal decomposition.

Lemma 3.5.2. *$\mathcal{V}^{-1}$ induces an isomorphism $C^{(p)} \to N/D$.*

Proof. Let $P = L \oplus T$ be a normal decomposition. Since C consists of classes of elements of L and D is the $\mathcal{F}$-image of T^σ, the isomorphism (3.4) shows that the composition

$$C^{(p)} \xrightarrow{\mathcal{V}^{-1}} N \longrightarrow N/D$$

is bijective. □

Thus from M we canonically obtain an F-zip $(N, C, D, \mathcal{F}, \mathcal{V}^{-1})$, which will be denoted by $M/I_R M$.

Next we consider the case that M is equipped with a principal quasi-polarization $\langle\ ,\ \rangle$, where a quasi-polarization is a non-degenerate alternating bilinear form $\langle\ ,\ \rangle : P \otimes_{W(R)} P \to W(R)$ such that

$$^\tau\langle \mathcal{V}^{-1}(1 \otimes x), \mathcal{V}^{-1}(1 \otimes y)\rangle = \langle x, y\rangle \tag{3.5}$$

for $x, y \in Q$, and it is called *principal* if the bilinear form is perfect.

The principal quasi-polarization $\langle\ ,\ \rangle$ induces a perfect alternating bilinear form $\langle\ ,\ \rangle : N \otimes_R N \to R$. The next lemma says that this is a polarization on $M/I_R M$.

Lemma 3.5.3. *(1) C and D are totally isotropic.*
(2) $\langle \dot\varphi y, \varphi x\rangle = \langle y, x\rangle^{(p)}$ for $x \in N^{(p)}$ and $y \in C^{(p)}$.

Proof. (1) For $x, y \in L$, we have $\langle x, y\rangle = {}^\tau\langle \mathcal{V}^{-1}x, \mathcal{V}^{-1}y\rangle \in I_R$. Since C is generated by classes of elements of L, we have $\langle C, C\rangle = 0$. For $x, y \in T$, we have

$$^\tau\langle \mathcal{F}(1 \otimes x), \mathcal{F}(1 \otimes y)\rangle = {}^\tau\langle \mathcal{V}^{-1}(1 \otimes {}^\tau 1x), \mathcal{V}^{-1}(1 \otimes {}^\tau 1y)\rangle = \langle {}^\tau 1x, {}^\tau 1y\rangle = ({}^\tau 1)^2\langle x, y\rangle.$$

Hence $\langle \mathcal{F}(1 \otimes x), \mathcal{F}(1 \otimes y)\rangle \in I_R$. Since D is the $\mathcal{F}$-image of T^σ in N, we have $\langle D, D\rangle = 0$. Since $\langle\ ,\ \rangle$ is perfect on N, both of C and D have to be totally isotropic.

(2) This follows immediately from the fact $\langle \mathcal{V}^{-1}(1 \otimes y), \mathcal{F}(1 \otimes x)\rangle = {}^\sigma\langle y, x\rangle$ for every $y \in Q$ and $x \in P$, see [29], (20) after Def. 18. □

Definition 3.5.4. Let R be as in Th. 2.1.2. Let X be a (principally quasi-polarized) formal p-divisible group over R and let M be the associated (principally quasi-polarized) display obtained by Th. 2.1.2. The *(polarized) F-zip of X* is defined to be $\mathrm{Fz}(X) := M/I_R M$.

4. Classifying data of F-zips

In this section let k denote an algebraically closed field of characteristic p. We recall the classification of (polarized) F-zips over k. Originally the classification of BT_1's is due to Kraft [16], and that of polarized BT_1's is due to Oort [23]. Now Moonen [20] and Moonen-Wedhorn [21] gave a more conceptual reinterpretation and a generalization by using Weyl groups.

Let G be a connected reductive group over k. Let $W = W_G$ be the Weyl group and I be its set of simple reflections. For a subset $J \subset I$, we denote by W_J the subgroup of W generated by the elements of J. Let ${}^J W$ be the set of $(J, \emptyset)$-reduced elements of W ([2], Chap. IV, Ex. §1, 3), which is a set of representatives of $W_J \backslash W$. We call an element of ${}^J W$ a *final element* of W with respect to J.

4.1. **The unpolarized case.** Let $G = \mathrm{GL}_h$. Let W be the Weyl group of G. We identify W and $\mathrm{Aut}(\{1,\dots,h\})$ in the usual sense. Note that W is generated by simple reflections $s_i = (i, i+1)$; write $I = \{s_1, \dots, s_{h-1}\}$. Let us explain the classification of F-zips over k of type (h_0, h_1) with $h_0 + h_1 = h$.

Theorem 4.1.1 ([21], (4.4))**.** *There is a canonical bijection*

$$\mathcal{E}^{\mathrm{un}}: \quad \{F\text{-zips over } k \text{ of type } (h_0,h_1)\}/\simeq \xrightarrow{\ \sim\ } {}^J W,$$

where $J = J_{(h_0,h_1)}$ is the parabolic type associated to (h_0,h_1); explicitly ${}^J W$ is described as $\{w \in W \mid w^{-1}(1) < \cdots < w^{-1}(h_1),\ w^{-1}(h_1+1) < \cdots < w^{-1}(h)\}$ (see [21] (1.9)*).*

There are some equivalent classifying data of F-zips. First in order to explain the inverse map of $\mathcal{E}^{\mathrm{un}}$, we introduce final types. A *final type of type* (h_0,h_1) is a pair (B,δ) consisting of a totally ordered finite set B and a map $\delta: B \to \{0,1\}$ with $h_* = \sharp\{b \mid \delta(b) = *\}$ for $* = 0,1$. For two final types $\mathcal{B} = (B,\delta)$ and $\mathcal{B}' = (B',\delta')$, we say $\mathcal{B}$ and $\mathcal{B}'$ are *isomorphic* if there exists a bijection f from B to B' preserving order such that $\delta = \delta' \circ f$. For a final type (B,δ), there exists a unique automorphism $\pi = \pi_\delta$ of B such that $\pi(b') > \pi(b) \Leftrightarrow \delta(b') > \delta(b)$ for any $b' < b$, see [11], Lem. 4.3 (1). To $w \in {}^J W$, we associate a final type $\mathcal{B} = (B,\delta)$ with $B = \{b_1 < \cdots < b_h\}$ defined by $\delta(b_i) = 1$ if $w(i) \le h_1$ and $\delta(b_i) = 0$ if $w(i) > h_1$. We have $\pi(b_i) = b_{h_0 + w(i)}$ for $\delta(b_i) = 1$ and $\pi(b_i) = b_{w(i)-h_1}$ for $\delta(b_i) = 0$.

For $w \in {}^J W$, we define an F-zip $Z_w = (N, C, D, \varphi, \dot\varphi)$ over $\mathbb{F}_p$ as follows. This gives the inverse map of $\mathcal{E}^{\mathrm{un}}$. Let $\mathcal{B} = (B,\delta)$ be the final type of w. Write $B = \{b_1 < \cdots < b_h\}$ and set $\pi = \pi_\delta$. First N is an $\mathbb{F}_p$-vector space with basis indexed by $b_1,\dots,b_h$, simply say $N = \bigoplus_{i=1}^h \mathbb{F}_p b_i$; and we define $C = \bigoplus_{\delta(b_i)=1} \mathbb{F}_p b_i$ and $D = \bigoplus_{\delta(b_i)=0} \mathbb{F}_p \pi(b_i)$ with φ and $\dot\varphi$ given by

$$\varphi(b_i) := \pi(b_i) \quad \text{if} \quad \delta(b_i) = 0,$$

and

$$\dot\varphi(b_i) := \begin{cases} \pi(b_i) & \text{if} \quad \delta(b_i) = 1,\ \delta(\pi(b_i)) = 1, \\ -\pi(b_i) & \text{if} \quad \delta(b_i) = 1,\ \delta(\pi(b_i)) = 0. \end{cases}$$

Definition 4.1.2. Let S' be an S-scheme. An F-zip Z over S is called *S'-split* of type w if Z is isomorphic to Z_w over S'. We define a BT_1 G_w over $\mathbb{F}_p$ by $\mathrm{fz}(G_w) = Z_w$.

We will use another classifying datum. A *final sequence of type* (h_0,h_1) is a map

$$\nu: \quad \{0,\dots,h\} \longrightarrow \{0,\dots,h_0\}$$

such that $\nu(0) = 0$ and $\nu(i-1) \le \nu(i) \le \nu(i-1)+1$ for $i = 1,\dots,h$. To $w \in {}^J W$, we associate a final sequence $\nu = \nu_w$ of type (h_0,h_1) defined by $\nu(i) = \sum_{j=1}^i (1-\delta(b_j))$, where $(\{b_j\},\delta)$ is the final type of w.

Note that the correspondences above give

$${}^J W \simeq \{\text{final types of type } (h_0,h_1)\}_{/\simeq} \simeq \{\text{final sequences of type } (h_0,h_1)\}.$$

4.2. **The polarized case.** Let $\mathbb{W} = \mathbb{W}_g$ be the Weyl group of Sp_{2g}. We can identify $\mathbb{W}$ in the usual way to

$$\mathbb{W} = \{w \in \mathrm{Aut}(\{1,\dots,2g\}) \mid w(i) + w(2g+1-i) = 2g+1\}. \tag{4.1}$$

Let I be the set of simple reflection $\{s_1,\dots,s_g\}$, where

$$s_i = \begin{cases} (i,i+1)\cdot(2g-i,2g+1-i) & \text{for } i<g, \\ (g,g+1) & \text{for } i=g. \end{cases} \tag{4.2}$$

Note that $\mathbb{W}$ is generated by I. Set $J = I \setminus \{s_g\}$. We know that $\mathbb{W}_J$ and ${}^J\mathbb{W}$ are given by

$$\mathbb{W}_J = \{w \in \mathbb{W} \mid w(\{1,\dots,g\}) = \{1,\dots,g\}\}, \tag{4.3}$$
$${}^J\mathbb{W} = \{w \in \mathbb{W} \mid w^{-1}(i) < w^{-1}(j) \text{ for any } 1 \le i < j \le g\}. \tag{4.4}$$

Theorem 4.2.1. *There is a canonical bijection*

$$\mathcal{E}: \quad \{\textit{polarized } F\textit{-zips over } k\}/\simeq \;\xrightarrow{\ \sim\ }\; {}^J\mathbb{W}.$$

Remark 4.2.2. In [23] Oort gave the classification in terms of polarized BT_1's and elementary sequences defined below. The description in 4.2.1 is found in Moonen-Wedhorn [21], (5.4); also see Moonen [20] for $p>2$.

Let $\mathcal{B} = (B,\delta)$ be a final type with $B = \{b_1 < \cdots < b_h\}$. The *dual final type* $\mathcal{B}^\vee = (B^\vee, \delta^\vee)$ is defined as $B^\vee = \{b_h^\vee < \cdots < b_1^\vee\}$ and $\delta^\vee(b_i^\vee) = 1 - \delta(b_i)$. Put $\pi = \pi_\delta$ and $\pi^\vee = \pi_{\delta^\vee}$. Then we have $\pi(b) = c$ if and only if $\pi^\vee(b^\vee) = c^\vee$. We say (B,δ) to be *symmetric* if (B,δ) is isomorphic to $(B^\vee,\delta^\vee)$. If $\mathcal{B}$ is symmetric, then h is even and $\mathcal{B}$ is of type (g,g) with $h=2g$.

To an element $w \in {}^J\mathbb{W}$, we associate a symmetric final type (B,δ) defined by $B = \{b_1 < \cdots < b_{2g}\}$ and $\delta(b_i) = 1$ if $w(i) \le g$ and $\delta(b_i) = 0$ if $w(i) > g$. Similarly to the unpolarized case, $\pi = \pi_\delta$ is given by $\pi(b_i) = b_{g+w(i)}$ for $\delta(b_i) = 1$ and $\pi(b_i) = b_{w(i)-g}$ for $\delta(b_i) = 0$.

For $w \in {}^J\mathbb{W}$, let $Z_w = (N, C, D, \varphi, \dot\varphi)$ be the F-zip defined as in §4.1. We define a polarization $\langle\ ,\ \rangle_w$ on Z_w by

$$\langle b_i, b_{2g+1-j}\rangle_w = \begin{cases} 1 & \text{if} \quad i=j \quad \text{and} \quad \delta(b_i)=0, \\ -1 & \text{if} \quad i=j \quad \text{and} \quad \delta(b_i)=1, \\ 0 & \text{if} \quad i \ne j. \end{cases}$$

Thus we have a polarized F-zip $(Z_w, \langle\ ,\ \rangle_w)$, which will be written simply as Z_w.

Definition 4.2.3. Let S' be an S-scheme. For $w \in \mathbb{W}_g$, a polarized F-zip Z over S is called *S'-split* of type w if Z is isomorphic to Z_w over S' as polarized F-zips. We define a polarized BT_1 G_w over $\mathbb{F}_p$ by $\mathrm{fz}(G_w) = Z_w$. The *local-local part of* w is the final element (of $\mathbb{W}_{g'}$ for some $g' \le g$) related to the local-local factor of G_w.

A *symmetric final sequence* of length $2g$ is a final sequence of type (g,g) of length $2g$:

$$\psi: \quad \{0,1\dots,2g\} \longrightarrow \{0,1,\dots,g\}$$

satisfying $\psi(2g-i) = g + \psi(i) - i$. An *elementary sequence* of length g is the restriction of a symmetric final sequence of length $2g$ to $\{1,\dots,g\}$. Clearly to give an elementary sequence of length g is equivalent to giving a symmetric final sequence of length $2g$. For $w \in {}^J\mathbb{W}$, we have a symmetric final sequence ψ_w defined by $\psi_w(i) = \sum_{j=1}^{i}(1-\delta(b_j))$.

The correspondences above give

$$^J\mathbb{W} \simeq \{\text{sym. final type of length } 2g\}_{/\simeq} \simeq \{\text{sym. final seq. of length } 2g\}.$$

4.3. **The Ekedahl-Oort stratification.** The main reference for the Ekedahl-Oort stratification is [23]. For a formulation in terms of Weyl groups, see [5], [6] and [20].

For $w \in {}^J\mathbb{W}$, the EO-stratum $\mathcal{S}_w$ is defined to be the subset of $\mathcal{A}_g$ consisting of points $y \in \mathcal{A}_g$ where y comes over some field from a principally polarized abelian variety A_y such that $\mathcal{E}(\mathrm{Fz}(A_y)) = w$, see [23], (5.11). As shown in [23], (3.2), $\mathcal{S}_w$ has a natural structure of a locally closed reduced subscheme of $\mathcal{A}_g$.

Here are fundamental results on the Ekedahl-Oort stratification:

Theorem 4.3.1 ([23]). *Let w be any element of ${}^J\mathbb{W}$.*

(1) *$\mathcal{S}_w$ is not empty.*
(2) *Every irreducible component of $\mathcal{S}_w$ has dimension $\ell(w)$, the length of w.*
(3) *$\mathcal{S}_w$ is quasi-affine for every $w \in {}^J\mathbb{W}$.*
(4) *$\mathcal{S}_{w'} \subset \overline{\mathcal{S}_w}$ is equivalent to $\mathcal{S}_{w'} \cap \overline{\mathcal{S}_w} \neq \emptyset$.*

Recently Wedhorn proved

Theorem 4.3.2 ([28]). *For any two $w, w' \in {}^J\mathbb{W}$, we have $\mathcal{S}_{w'} \subset \overline{\mathcal{S}_w}$ if and only if there exists an element u of $\mathbb{W}_J$ such that $u^{-1} \cdot w' \cdot (w_{0,J} \cdot u \cdot w_{0,J}) \leq w$ with respect to the Bruhat-Chevalley order $\leq$. Here $w_{0,J}$ is the element of $\mathbb{W}_J$ sending i to $g+1-i$ for any $i = 1, \dots, g$.*

Remark 4.3.3. For $w \in \mathbb{W}$ and $1 \leq i, j \leq 2g$, we define $r_w(i,j) := \sharp\{a \leq i \mid w(a) \leq j\}$. It is known (cf. [5], §2.1 and [1], §3.3) that the Bruhat-Chevalley order is described as follows: for $w, w' \in \mathbb{W}$ we have $w' \leq w \Leftrightarrow r_{w'}(i,j) \geq r_w(i,j)$ for all $1 \leq i, j \leq 2g$.

Recall the result of Ekedahl and van der Geer:

Theorem 4.3.4 ([5], Th. 11.5). *Let $w \in {}^J\mathbb{W}$. If $\psi_w(\lfloor (g+1)/2 \rfloor) \neq 0$, then $\mathcal{S}_w$ is irreducible.*

Remark 4.3.5. Note that $\psi_w(\lfloor (g+1)/2 \rfloor) = 0$ if and only if $\mathcal{S}_w$ is contained in the supersingular locus, see [3], (4.8), Step 2. Also see [9] for another proof and a generalization.

Definition 4.3.6. Let $\xi(w)$ denote the Newton polygon of the generic point of $\mathcal{S}_w$ if $\mathcal{S}_w$ is not contained in the supersingular locus and otherwise the supersingular Newton polygon. We call $\xi(w)$ the *generic Newton polygon* of $\mathcal{S}_w$.

5. Foliations

We recall some known facts on the foliations (central leaves and isogeny leaves) and prove some new results we shall use later.

5.1. **Minimal p-divisible groups.** Firstly we review the theory of minimal p-divisible groups [25].

Definition 5.1.1. For non-negative integers m, n with $\gcd(m,n) = 1$, we define a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$P_{m,n} := \mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p e_i \tag{5.1}$$

with $\mathcal{F}, \mathcal{V}$ operations:

$$\mathcal{F}e_i = e_{i+n} \quad \text{and} \quad \mathcal{V}e_i = e_{i+m} \quad \text{for} \quad \forall i \in \mathbb{Z}_{\geq 0}, \tag{5.2}$$

where e_i $(i \in \mathbb{Z}_{\geq m+n})$ are defined as satisfying $e_{i+m+n} = pe_i$ for $i \in \mathbb{Z}_{\geq 0}$.

Let ϑ be the endomorphism defined by $\vartheta(x_i) = x_{i+1}$; then we have

$$\mathrm{End}_{\mathbb{F}_p}(P_{m,n}) = \mathbb{Z}_p[\vartheta]/(\vartheta^{m+n} - p).$$

Let θ denote the endomorphism of $H_{m,n}$ corresponding to ϑ.

For an arbitrary perfect field K, the Dieudonné module $P_{m,n,K} = \mathbb{D}(H_{m,n} \otimes K)$ has a $W(K)$-basis $\{e_0, \dots, e_{m+n-1}\}$ satisfying the equations (5.2), which is called a *minimal basis* of $P_{m,n,K}$; and e_0 (resp. e_{m+n-1}) is called the *highest* (resp. *lowest*) element.

For a Newton polygon $\xi = \sum_{l=1}^{\mathfrak{t}} (m_l, n_l)$, we write

$$H(\xi) = \bigoplus_{l=1}^{\mathfrak{t}} H_{m_l,n_l} \qquad \text{and} \qquad P(\xi) = \bigoplus_{l=1}^{\mathfrak{t}} P_{m_l,n_l}. \tag{5.3}$$

Note that the Newton polygon of $H(\xi)$ is equal to ξ.

Definition 5.1.2. A p-divisible group X is called *minimal* if there exist a Newton polygon ξ and an isomorphism from X to $H(\xi)$ over an algebraically closed field. If a BT_1 G is isomorphic to $H(\xi)[p]$ over an algebraically closed field, we call G *minimal*, and also the F-zip $\mathrm{fz}(G)$ and the final element of G are called *minimal.*

5.2. **Central leaves.** Let k be an algebraically closed field of characteristic p. Let $(X, \imath)$ be a principally quasi-polarized p-divisible group over k. The *central leaf* for $(X, \imath)$ is defined by

$$\mathcal{C}_{(X,\imath)} = \{(A, \eta) \in \mathcal{A}_g \mid (A[p^\infty], \eta[p^\infty])_\Omega \simeq (X, \imath)_\Omega \text{ over some alg. closed field } \Omega\}.$$

For a geometric point $x \in \mathcal{A}_g$, let (A, η) be the associated principally polarized abelian variety; we set $\mathcal{C}_x := \mathcal{C}_{(A[p^\infty],\eta[p^\infty])}$. In [24], (3.3), it was proved that $\mathcal{C}_x$ is closed in $\mathcal{W}_\xi^0$ with $\xi = \mathcal{N}(A)$; we consider this is a closed subscheme by giving it the induced reduced scheme structure.

The next proposition says $\mathcal{C}_{(X,\imath)} \neq \emptyset$ for any principally quasi-polarized p-divisible group $(X, \imath)$. This result and the proof below are due to Oort (private communication).

Proposition 5.2.1. *Let $(X, \imath)$ be a principally quasi-polarized p-divisible group over k. Then there exists a principally polarized abelian variety (A, η) over k such that $(A[p^\infty], \eta[p^\infty]) \simeq (X, \iota)$.*

To prove this, we need a lemma:

Lemma 5.2.2. *Let ξ be a symmetric Newton polygon. Let $\zeta^{(1)}$ and $\zeta^{(2)}$ be two quasi-polarizations on $H(\xi)_k$. For a sufficient large $n \geq 0$, we have $(p^n)^*\zeta^{(1)} = u^*\zeta^{(2)}$ for a certain isogeny $u : H(\xi)_k \to H(\xi)_k$.*

Proof. Let $\mathbf{I}_r$ and $\mathbf{II}_r$ be the quasi-polarizations on $H_{1,1}$ and $H_{1,1} \oplus H_{1,1}$ respectively defined in [24, 3.5] (also see [17, 6.1]), and let $\zeta_d(m, n)$ be the quasi-polarization on $H_{m,n} \oplus H_{n,m}$ defined in [24, 3.6]. Note that $p^*\mathbf{I}_r = \mathbf{I}_{r+2}$ and $p^*\mathbf{II}_r = \mathbf{II}_{r+2}$, and also $p^*\zeta_d(m, n) = \zeta_{d+2(m+n)}(m, n)$.

Write $\xi = s(1,1) + \sum_i \{(m_i, n_i) + (m_i, n_i)\}$ with $m_i > n_i$ and $\gcd(m_i, n_i) = 1$. By [24, 3.7], $\zeta^{(*)}$ for $* = 1, 2$ is isomorphic to $\zeta^{(*)}(s,s) \oplus \bigoplus_i \zeta_{d_i^{(*)}}(m_i, n_i)$, where the first factor is a direct sum of some quasi-polarizations of types $\mathbf{I}_r$ and $\mathbf{II}_r$ ($r \in \mathbb{Z}_{\geq 0}$).

Hence it suffices to show the supersingular case and the case of $\xi = (m,n)+(n,m)$ for $m > n$ with $\gcd(m,n) = 1$. For the supersingular case, the lemma follows from the fact that $F^*\mathbf{I}_r = \mathbf{I}_{r+1}$ and $F^*\mathbf{II}_r = \mathbf{II}_{r+1}$ and the fact that $u^*(\mathbf{I}_r \oplus \mathbf{I}_r) = \mathbf{II}_r$, where u is defined as follows: note that $\mathbf{I}_r \oplus \mathbf{I}_r$ is isomorphic to the quasi-polarization defined by $\langle x, \mathcal{F}y\rangle = p^r$ and $\langle x, \mathcal{F}x\rangle = \langle y, \mathcal{F}y\rangle = \langle x, y\rangle = 0$ on $P_{1,1}x \oplus P_{1,1}y$ ([17], §6.1, Remark); then u is the isogeny corresponding to the inclusion

$$P_{1,1}x \oplus P_{1,1}\mathcal{F}y \subset P_{1,1}x \oplus P_{1,1}y.$$

The case $\xi = (m,n) + (n,m)$ follows from the fact that $v^*\zeta_d(m,n) = \zeta_{d+1}(m,n)$, where v is the isogeny $\theta \oplus \mathrm{id} : H_{m,n} \oplus H_{n,m} \to H_{m,n} \oplus H_{n,m}$. □

Proof of Prop. 5.2.1. Put $\xi = \mathcal{N}(X)$. Since $\mathcal{W}^0_\xi \neq \emptyset$, there exists a principally polarized abelian variety (A_1, η_1) over k such that $A_1[p^\infty]$ is isogenous to X over k. There exists an abelian variety A_2 over k with an isogeny $f : A_2 \to A_1$ such that $A_2[p^\infty]$ is minimal and $\deg(f)$ is a power of p. We have a polarization $\eta_2 := f^*\eta_1$ on A_2. Choose an isogeny $v : Y \to X$ with Y minimal and set $\jmath = v^*\imath$. By Lem. 5.2.2, replacing f by $f \circ p^n : A_2 \to A_1$ for sufficient large n, we may assume that $\eta_2[p^\infty] = u^*\jmath$ for a certain isogeny $u : A_2[p^\infty] \to Y$. Note that $\deg(\eta_2) = \deg(v)^2 \deg(u)^2$ and this is a power of p. Let $G := \mathrm{Ker}(v \circ u) \subset A_2$ and set $A = A_2/G$. Since G is isotropic, i.e., $\eta_2(G) = 0$, it follows from [22, Cor. on p. 231] that η_2 descends to a polarization η on A; clearly $\deg(\eta) = 1$. □

5.3. **Central streams.** Let ξ be a symmetric Newton polygon. By [24], Prop. 3.7, there exists a principal quasi-polarization $\imath$ on $H(\xi)$, which is unique up to isomorphism of $H(\xi)$. Thus we have a central leaf

$$\mathcal{Z}_\xi = \mathcal{C}_{(H(\xi),\imath)}.$$

We call $\mathcal{Z}_\xi$ the *central stream* of the Newton polygon ξ.

Theorem 5.3.1 (Oort, [25])**.** *Let X be a p-divisible group over an algebraically closed field Ω. If $X[p] \simeq H(\xi)[p] \otimes \Omega$, then $X \simeq H(\xi) \otimes \Omega$.*

Let w_ξ be the element of ${}^J\mathbb{W}$ corresponding to $(H(\xi)[p], \imath[p])$. Then Th. 5.3.1 implies

$$\mathcal{Z}_\xi = \mathcal{S}_{w_\xi}. \tag{5.4}$$

By Th. 4.3.4, $\mathcal{Z}_\xi$ is irreducible if ξ is not supersingular.

5.4. **Isogeny leaves.** Let k be an algebraically closed field. Let $x \in \mathcal{W}^0_\xi(k)$. Oort defined the isogeny leaf $\mathcal{I}_x$ in $\mathcal{W}^0_\xi$, see [24], (4.2), and showed that $\mathcal{I}_x$ is closed in $\mathcal{W}^0_\xi$ and proper over k, see [24], (4.11).

Let R be an integral domain of finite type over k with $\dim(R) \geq 1$ and let $\mathfrak{m}$ be a maximal ideal of R with $R/\mathfrak{m} = k$. Let $\mathcal{X}$ be a principally quasi-polarized p-divisible group over R with $\mathcal{X} \otimes (R/\mathfrak{m}) \simeq A_x[p^\infty]$. Assume we are given a non-trivial family over R of isogenies as polarized p-divisible groups

$$\rho : \quad (H(\xi), \zeta) \otimes R \longrightarrow \mathcal{X}. \tag{5.5}$$

Let A_1 be a polarized abelian variety over k with isogeny $\tilde\rho : A_1 \to A_x$ such that $A_1[p^\infty] \simeq (H(\xi), \zeta)_k$ and $\tilde\rho[p^\infty] \simeq \rho \otimes (R/\mathfrak{m})$. Set $G = \mathrm{Ker}\,\rho$. Then we have a principally polarized abelian scheme $A = A_{1,R}/G$ over R (cf. [22, Cor. on p. 231]). Let T be the image of the induced morphism $\mathrm{Spec}(R) \to \mathcal{A}_g$.

Lemma 5.4.1. *$T \subset \mathcal{I}_x$ and $\dim(T) > 0$.*

Proof. By definition T is an H_α-subscheme in $\mathcal{A}_g$, see [24], (4.1). Hence $T \subset \mathcal{I}_x$. Since ρ is non-trivial, $\dim(T) > 0$ follows from the rigidity of homomorphisms of p-divisible groups (cf. [29], Prop. 40). □

6. Strategy

Now we explain how to prove the main theorem in § 1.

6.1. **Reduction of the problem.** Let k be an algebraically closed field of characteristic p. In this subsection we prove that the main theorem follows from

Theorem 6.1.1. *Assume $w \in {}^J\mathbb{W}$ is not minimal. There exists a principally quasi-polarized p-divisible group $\mathcal{X}$ over a positive dimensional irreducible scheme S of finite type over k such that*

(1) *there is a non-trivial family of isogenies of quasi-polarized p-divisible groups:*

$$(H(\xi(w)), \zeta) \times S \longrightarrow \mathcal{X}$$

for a certain quasi-polarization ζ on $H(\xi(w))$, and

(2) *$\mathcal{X}$ is decomposed as $X_{ét} \oplus \mathcal{Y} \oplus X_{ét}^t$ with an étale p-divisible group $X_{ét}$ over S and we have $\mathrm{Fz}(\mathcal{Y}) \simeq Z_{\overline{w}} \times S$ (see Def. 3.5.4 for the definition of $\mathrm{Fz}(\mathcal{Y})$), where $\overline{w}$ is the local-local part of w (see Def. 4.2.3).*

The proof will occupy the rest of sections.

Remark 6.1.2. This theorem can be seen as a complement to Oort's theorem [25] (see Th. 5.3.1 above). His theorem implies that if w is minimal, then there is no such a family as in Th. 6.1.1. We also mention a relation to [26], (8.1), where Oort constructed, for any non-minimal w, a positive dimensional non-trivial family of p-divisible groups with p-kernel type w and with constant Newton polygon which is the same as that of $\mathcal{L}(G_w)$, where $\mathcal{L}(G_w)$ is the p-divisible group introduced in [26], (2.5) (called the standard lift of G_w). However the Newton polygon of $\mathcal{L}(G_w)$ is not always equal to $\xi(w)$ (e.g. $w = (1, g+1, \dots, 2g-1; 2, \dots, g, 2g) \in {}^J\mathbb{W}$ for $g \geq 3$) and also [26] takes no account of quasi-polarizations.

Here is a corollary:

Corollary 6.1.3. *Assume $w \in {}^J\mathbb{W}$ is not minimal. Then for every geometric point x of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$, a component of $\mathcal{I}_x \cap \mathcal{S}_w$ has dimension > 0.*

Proof. By Th. 6.1.1, Prop. 5.2.1 and Lem. 5.4.1, there exists a geometric point y of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$ such that a component of $\mathcal{I}_y \cap \mathcal{S}_w$ has dimension > 0. Note that $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$ is open dense in $\mathcal{S}_w$ and therefore is regular (as a stack) because $\mathcal{S}_w$ is so. Let x be any geometric point of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$. By definition the central leaf $\mathcal{C}_x$ is contained in $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$. Since a component of $\mathcal{I}_y \cap \mathcal{S}_w$ has dimension > 0, we have $\dim \mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w > \dim \mathcal{C}_x (= \dim \mathcal{C}_y)$; then [24], Th. 5.3 shows the corollary. □

Using the corollary, we can prove the main theorem.

Proof of (Cor. 6.1.3 $\Rightarrow$ Main theorem). If w is minimal, then $\mathcal{Z}_{\xi(w)} = \mathcal{S}_w$; hence the main theorem is obviously true. Assume w is not minimal. Assume the main theorem is true for all w' with $\mathcal{S}_{w'} \subsetneq \overline{\mathcal{S}_w}$. (The smallest case w.r.t. $\subset$ is the superspecial case $w = \mathrm{id}$ and in this case w is minimal.) According to Cor. 6.1.3 there exists a geometric point x of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$ such that a component of $\mathcal{I}_x \cap \mathcal{S}_w$ has dimension > 0. Since $\mathcal{I}_x$ is proper and $\mathcal{S}_w$ is quasi-affine, there exists w' with $S_{w'} \subsetneq \overline{S_w}$ such that we have $\mathcal{S}_{w'} \cap \mathcal{I}_x \neq \emptyset$. Clearly $\mathcal{S}_{w'} \subset \overline{\mathcal{S}_w}$ implies $\xi(w') \prec \xi(w)$, and from $\mathcal{I}_x \subset \mathcal{W}^0_{\xi(w)}$ and $\mathcal{S}_{w'} \cap \mathcal{I}_x \neq \emptyset$ we have $\xi(w) \prec \xi(w')$; hence we obtain $\xi(w) = \xi(w')$. By the hypothesis of induction, we have $\mathcal{Z}_{\xi(w')} \subset \overline{\mathcal{S}_{w'}}$. Then $\mathcal{Z}_{\xi(w)} = \mathcal{Z}_{\xi(w')} \subset \overline{\mathcal{S}_{w'}} \subset \overline{\mathcal{S}_w}$. $\square$

6.2. **Outline of the proof of Th. 6.1.1.** Let us explain the strategy of our proof of Th. 6.1.1. Let $w \in {}^J\mathbb{W}$ and assume w is not minimal. If $\xi(w)$ is supersingular, $\mathcal{C}_x$ consists of points for any $x \in \mathcal{W}_{\xi(w)}$; then $\mathcal{I}_x \cap \mathcal{S}_w$ is positive dimensional (because $w \neq \mathrm{id}$); hence there is nothing to prove; from now on we assume $\xi(w)$ is not supersingular. Write

$$\xi(w) = \sum_{l=1}^{\mathfrak{t}} (m_l, n_l)$$

with $\lambda_1 \leq \cdots \leq \lambda_{\mathfrak{t}}$, where $\lambda_l = m_l/(m_l + n_l)$. Put $(d, c) = (m_1, n_1) = (n_{\mathfrak{t}}, m_{\mathfrak{t}})$. Since $\xi(w)$ is not supersingular, we have $\mathfrak{t} \geq 2$ and $c > d$.

Take a geometric point $x : \mathrm{Spec}(k) \to \mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$. Let (A, η) be the principally polarized abelian variety at x and set $X = A[p^\infty]$. From the composition of an embedding $\iota : \mathbb{D}(X) \to M(\xi(w))_k$ and the natural projection $\mathrm{pr} : M(\xi(w))_k \to M_{d,c,k}$, we have a homomorphism $X \to X_1$, where X_1 is the p-divisible group corresponding to the image of $\mathrm{pr} \circ \iota$. The homomorphism $X \to X_1$ makes a self-dual complex over k:

$$0 \longrightarrow X_1^t \longrightarrow X \longrightarrow X_1 \longrightarrow 0. \tag{6.1}$$

This induces a self-dual complex of F-zips over k:

$$\mathcal{C}_0^\bullet : \quad 0 \longrightarrow Z_1^\vee \xrightarrow{f_0^\vee} Z \xrightarrow{f_0} Z_1 \longrightarrow 0. \tag{6.2}$$

The proof consists of four steps. The first step is to prove that X_1 is a minimal p-divisible group, see the next subsection (Prop. 6.3.1). This is necessary for the remaining steps. As the second step we shall extend $\mathcal{C}_0^\bullet$ to a non-trivial self-dual complex of F-zips

$$\mathcal{C}^\bullet : \quad 0 \longrightarrow Z_{1,S}^\vee \xrightarrow{f^\vee} Z_S \xrightarrow{f} Z_{1,S} \longrightarrow 0 \tag{6.3}$$

over some positive-dimensional smooth scheme S over k (see Prop. 7.6.1 for more precise statement. We remark that the F-zips are constant and only the homomorphism moves). The third step is to extend $\mathcal{C}^\bullet$ to a self-dual complex of p-divisible groups

$$\mathcal{D}^\bullet : \quad 0 \longrightarrow X_{1,S'}^t \longrightarrow \mathcal{X} \longrightarrow X_{1,S'} \longrightarrow 0 \tag{6.4}$$

after some base extension $S' \to S$. This is done in Prop. 8.3.1. Finally, based on this construction of $\mathcal{X}$ from $\mathcal{C}_0^\bullet$, we find a family required in Th. 6.1.1 (see §8.4).

6.3. **Minimality of X_1.** Let X_1 be as in §6.2. We prove

Proposition 6.3.1. *X_1 is isomorphic to $H_{d,c} \otimes k$.*

For this, we recall the results of [9] and [10] on the optimal upper bound of the last Newton slopes of (principally quasi-polarized) p-divisible groups with given isomorphism type of p-kernel.

Let ν be a final sequence of length h. We define

$$\Psi: \quad \{1,\dots,h\} \longrightarrow \{1,\dots,h\}$$

by sending i to $\nu(i)$ if $\nu(i) \neq 0$ and $\nu(h)+i$ if $\nu(i)=0$. We get a non-empty subset

$$\Sigma := \bigcap_{j=1}^{\infty} \operatorname{Im} \Psi^j$$

of the set $\{1,\dots,h\}$. Set $\Sigma' := \Sigma \cap \{1,2,\dots,\nu(h)\}$. Then we define

$$\rho_\nu = \sharp\Sigma'/\sharp\Sigma. \tag{6.5}$$

Theorem 6.3.2 ([9])**.** *Let $w \in {}^J\mathbb{W}$ and let ν be the (symmetric) final sequence ψ_w of w. Then the last slope of $\xi(w)$ is equal to ρ_ν.*

Next we recall an unpolarized analogue of Th. 6.3.2. Let G_ν be a BT_1 over $\mathbb{F}_p$ with final sequence ν.

Theorem 6.3.3 ([10], Cor. 1.3 and 5.4)**.** (1) *The optimal upper bound of the last Newton slopes of p-divisible groups with given final sequence ν is equal to ρ_ν.*

(2) $\rho_\nu = \max\{m/(m+n) \mid H_{m,n}[p]_\Omega \overset{\exists}{\hookrightarrow} G_{\nu,\Omega}$ *for some alg. closed field* $\Omega\}$.

Proof of Prop. 6.3.1. It suffices to prove that the final sequence of $X_1^t[p]$ is $\nu_{c,d}$. Let ν be the (symmetric) final sequence of $X[p]$. By the construction of X, the last slope of $\xi(w)$ is ρ_ν, i.e., $\rho_\nu = c/(c+d)$. Let ν' be the final sequence of $X_1^t[p]$. Since $X_1^t[p] \hookrightarrow X[p]$, i.e., $G_{\nu',k} \hookrightarrow G_{\nu,k}$, we have $\rho_{\nu'} \leq \rho_\nu$ by Th. 6.3.2 and Th. 6.3.3 (2). By the construction of X_1, the (last) Newton slope of X_1^t is ρ_ν; hence we have $\rho_\nu \leq \rho_{\nu'}$ by Th. 6.3.3 (1) for ν'. Thus $\rho_{\nu'} = \rho_\nu$. Then Th. 6.3.3 (2) implies that there exists an injection $H_{c,d}[p]_\Omega \hookrightarrow G_{\nu',\Omega}$ for some $\Omega = \overline{\Omega}$. Since $H_{c,d}[p]$ and $G_{\nu'}$ have the same rank ($= c+d$), we obtain $H_{c,d}[p]_\Omega \simeq G_{\nu',\Omega}$, namely $\nu_{c,d} = \nu'$. □

7. The space of homomorphisms of F-zips

The aim of this section is to prove Prop. 7.6.1, where we construct a non-trivial family of complexes of F-zips as in (6.3). For this, we start with describing the space of homomorphisms between F-zips.

7.1. **Slices and strings.** It is known (see [26], §2 and also [20], §4) that every homomorphism of F-zips can be described in terms of slices and strings. We write here the definition of slices and strings by making use of final types.

Definition 7.1.1. Let $\mathcal{B}_1 = (B_1, \delta_1)$ and $\mathcal{B}_2 = (B_2, \delta_2)$ be final types and set $\pi_1 = \pi_{\delta_1}$ and $\pi_2 = \pi_{\delta_2}$.

(1) A *finite slice* ω is a subset of $B_1 \times B_2$ of the form

$$\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\} \qquad \text{with} \quad |\omega| = \ell \tag{7.1}$$

for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying

(a) $\delta_1(s_1) = 1$ and $\delta_2(s_2) = 0$,
(b) $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \leq i < \ell$ and
(c) $\delta_1(\pi_1^\ell(s_1)) = 0$ and $\delta_2(\pi_2^\ell(s_2)) = 1$.

We denote by $\Omega_{\mathrm{fin}} = \Omega_{\mathrm{fin}}(\mathcal{B}_1, \mathcal{B}_2)$ the set of finite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$.

(2) An *infinite slice* ω is a subset of $B_1 \times B_2$ of the form

$$\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\} \qquad \text{with} \quad |\omega| = \ell \tag{7.2}$$

for $s_1 \in B_1$ and $s_2 \in B_2$ satisfying
(a) $s_1 = \pi_1^\ell(s_1)$ and $s_2 = \pi_2^\ell(s_2)$,
(b) $\delta_1(\pi_1^i(s_1)) = \delta_2(\pi_2^i(s_2))$ for all $1 \leq i < \ell$.

We denote by $\Omega_\infty = \Omega_\infty(\mathcal{B}_1, \mathcal{B}_2)$ the set of infinite slices of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Set $\Omega = \Omega(\mathcal{B}_1, \mathcal{B}_2) := \Omega_{\mathrm{fin}} \sqcup \Omega_\infty$.

To a slice ω, we associate the subgroup scheme $\mathbb{K}_\omega$ of the additive group $\mathbb{G}_a$ over $\mathbb{F}_p$ defined by

$$\mathbb{K}_\omega = \begin{cases} \mathbb{G}_a & \text{if } \omega \in \Omega_{\mathrm{fin}}, \\ \mathrm{Ker}(F^{|\omega|} - \mathrm{id} : \ \mathbb{G}_a \to \mathbb{G}_a) & \text{if } \omega \in \Omega_\infty. \end{cases}$$

Let S be an $\mathbb{F}_p$-scheme. Let $\omega = \{(\pi_1^i(s_1), \pi_2^i(s_2)) \mid 1 \leq i \leq \ell\}$ be a slice with $|\omega| = \ell$. For an element $r \in \omega$, we denote by $\varepsilon(r)$ $(= \varepsilon_\omega(r))$ the integer ε with $0 \leq \varepsilon < \ell$ satisfying $r = (\pi_1^{\varepsilon+1}(s_1), \pi_2^{\varepsilon+1}(s_2))$. For $a \in \mathbb{K}_\omega(S)$, we define a map

$$\mathrm{st}_{\omega,a} : \quad B_1 \times B_2 \longrightarrow \mathbb{K}_\omega(S) \tag{7.3}$$

by sending $r \in \omega$ to $a^{p^{\varepsilon(r)}}$ and $r \notin \omega$ to 0.

Lemma 7.1.2. *Let w_1 and w_2 be elements of ${}^{J_1}W_{\mathrm{GL}_{h_1}}$ and ${}^{J_2}W_{\mathrm{GL}_{h_2}}$ respectively. Let Z_1 and Z_2 be the split F-zips over $\mathbb{F}_p$ of type w_1 and w_2 respectively. Then the functor, from the category of $\mathbb{F}_p$-schemes to the category of sets, sending S to $Hom_S(Z_{1,S}, Z_{2,S})$ is represented by a scheme $\mathrm{Hom}(Z_1, Z_2)$ over $\mathbb{F}_p$, which has a canonical commutative group scheme structure. Moreover there is an isomorphism as group schemes over $\mathbb{F}_p$:*

$$\Phi : \quad \bigoplus_{\omega \in \Omega} \mathbb{K}_\omega \xrightarrow{\ \sim\ } \mathrm{Hom}(Z_1, Z_2). \tag{7.4}$$

Proof. For $* = 1, 2$, let $\mathcal{B}_*$ be the final types of w_*. We write $\mathcal{B}_* = (B_*, \delta_*)$ with $B_* = \{b_1^{(*)} < \cdots < b_{h_*}^{(*)}\}$. Set $\pi_* = \pi_{\delta_*}$ and define $\varpi_*(i)$ $(1 \leq i \leq h_*)$ by $\pi_*(b_i) = b_{\varpi_*(i)}$. Also write $Z_* = (N_*, C_*, D_*, \varphi_*, \dot\varphi_*^{-1})$ with $N_* = \bigoplus_{i=1}^{h_*} \mathbb{F}_p b_i^{(*)}$ as defined in § 4.1. Let S be any $\mathbb{F}_p$-scheme. An $\mathcal{O}_S$-homomorphism $\mu : N_{1,S} \to N_{2,S}$, say

$$\mu(b_i^{(1)}) = \sum_j r_{i,j} b_j^{(2)} \qquad \text{with} \quad r_{i,j} \in \Gamma(S, \mathcal{O}_S),$$

gives an element of $Hom_S(Z_{1,S}, Z_{2,S})$ if and only if

$$\begin{cases} r_{i,j} = 0 & \text{if } \delta(b_i^{(1)}) = 1 \text{ and } \delta(b_i^{(2)}) = 0, \\ r_{\varpi_1(i), \varpi_2(j)} = 0 & \text{if } \delta(b_i^{(1)}) = 0 \text{ and } \delta(b_i^{(2)}) = 1, \\ r_{\varpi_1(i), \varpi_2(j)} = r_{i,j}^p & \text{if } r_{i,j} \neq 0 \text{ and } r_{\varpi_1(i), \varpi_2(j)} \neq 0. \end{cases} \tag{7.5}$$

Here note that the first equation is a paraphrase of $\mu(C_1) \subset C_2$ and the second is a paraphrase of $\mu(D_1) \subset D_2$, and the third is a paraphrase of $\mu \circ \varphi_1 = \varphi_2 \circ \mu^{(p)}$ or $\mu \circ \dot{\varphi}_1 = \dot{\varphi}_2 \circ \mu^{(p)}$. Clearly (7.5) is equivalent to that $r_{i,j}$ is of the form

$$\sum_{\omega \in \Omega} \mathrm{st}_{\omega,a}(b_{i,j})$$

for a certain $a \in \mathbb{K}_\omega(S)$, where $b_{i,j} = (b_i^{(1)}, b_j^{(2)}) \in B_1 \times B_2$. □

Definition 7.1.3. We retain the notation of Lem. 7.1.2. Let pr_ω be the projection $\bigoplus \mathbb{K}_\omega \to \mathbb{K}_\omega$. Let $f : Z_{1,S} \to Z_{2,S}$ be a homomorphism of F-zips. For a slice ω, the element $\mathrm{pr}_\omega \circ \Phi^{-1}(f)$ of $\mathbb{G}_a(S)$ is called the *string of f at ω*. An element of $\{\omega \in \Omega(\mathcal{B}_1, \mathcal{B}_2) \mid \mathrm{pr}_\omega \circ \Phi^{-1}(f) \neq 0\}$ is said to be *one of the slices defining f* or simply a *slice defining f*.

7.2. **Duality.** Let Z be an F-zip and let $\mathcal{B} = (B, \delta)$ be the final type of Z. Then the final type of $Z^\vee$ (cf. §3.3) is canonically $\mathcal{B}^\vee = (B^\vee, \delta^\vee)$ (cf. §4.2).

Let $N_1, N_2, \mathcal{B}_1$ and $\mathcal{B}_2$ be as in § 7.1. Let ω be a slice $\in \Omega(\mathcal{B}_1, \mathcal{B}_2)$. We define $\omega^\vee \in \Omega(\mathcal{B}_2^\vee, \mathcal{B}_1^\vee)$ by

$$\omega^\vee = \{(b_2^\vee, b_1^\vee) \mid (b_1, b_2) \in \omega\}.$$

Let Z_1 and Z_2 be as in Lem. 7.1.2. Clearly we have a commutative diagram:

$$\begin{array}{ccc} \displaystyle\bigoplus_{\omega \in \Omega(\mathcal{B}_1, \mathcal{B}_2)} \mathbb{K}_\omega & \longrightarrow & \displaystyle\bigoplus_{\omega^\vee \in \Omega(\mathcal{B}_2^\vee, \mathcal{B}_1^\vee)} \mathbb{K}_{\omega^\vee} \\ \downarrow & & \downarrow \\ \vee : \quad \mathrm{Hom}(Z_1, Z_2) & \longrightarrow & \mathrm{Hom}(Z_2^\vee, Z_1^\vee), \end{array}$$

where the vertical maps are obtained in Lem. 7.1.2 and the top horizontal map sends $a \in \mathbb{K}_\omega$ to $a \in \mathbb{K}_{\omega^\vee}$. Here we note $\mathbb{K}_\omega = \mathbb{K}_{\omega^\vee}$.

7.3. **Top and bottom elements.** As introduced in [20], 4.14, for a final type $\mathcal{B} = (B, \delta)$ we define the set $\mathrm{Top}(\mathcal{B})$ of *top elements* and the set $\mathrm{Bot}(\mathcal{B})$ of *bottom elements* by

$$\begin{array}{rcl} \mathrm{Top}(\mathcal{B}) & = & \{\mathfrak{t} \in B \mid \delta(\pi^{-1}(\mathfrak{t})) = 1,\ \delta(\mathfrak{t}) = 0\}, \\ \mathrm{Bot}(\mathcal{B}) & = & \{\mathfrak{b} \in B \mid \delta(\pi^{-1}(\mathfrak{b})) = 0,\ \delta(\mathfrak{b}) = 1\}. \end{array}$$

(See [14], 6.7 for a similar notion in the combinatorics of semi-modules.) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be final types. For any $(\mathfrak{t}, \mathfrak{b}) \in \mathrm{Top}(\mathcal{B}_1) \times \mathrm{Bot}(\mathcal{B}_2)$, we set $\omega_{\mathfrak{t},\mathfrak{b}} := \{(\mathfrak{t}, \mathfrak{b})\}$. Then obviously we have $\omega_{\mathfrak{t},\mathfrak{b}} \in \Omega_{\mathrm{fin}}(\mathcal{B}_1, \mathcal{B}_2)$.

Let m, n be coprime non-negative integers and let $\mathcal{B}_{m,n} = (B_{m,n}, \delta_{m,n})$ be the final type of the minimal BT_1 $H_{m,n}[p]$. If we write $B_{m,n} = \{b_1 < \cdots < b_{m+n}\}$, then we have $\delta_{m,n}(b_i) = 1$ for $1 \leq i \leq n$ and $\delta_{m,n}(b_i) = 0$ for $n < i \leq m+n$ (cf. [11], §4.5). Let $\pi_{m,n}$ be the automorphism of $B_{m,n}$ associated with $\delta_{m,n}$. Then we have the commutative diagram

$$\begin{array}{ccc} B_{m,n} & \longrightarrow & \mathbb{Z}/(m+n)\mathbb{Z} \\ {\scriptstyle \pi_{m,n}}\downarrow & & \downarrow{\scriptstyle +m} \\ B_{m,n} & \longrightarrow & \mathbb{Z}/(m+n)\mathbb{Z}, \end{array} \tag{7.6}$$

where the horizontal maps send b_i to the class of $i - 1$.

Lemma 7.3.1. *For any $l \in \mathbb{Z}_{\geq 0}$ we have*

$$\delta_{m,n}(\pi^i_{m,n}(b_1)) = \begin{cases} 1 & \text{for} \quad \lceil \frac{m+n}{m} l \rceil \leq i < \lceil \frac{m+n}{m} l + \frac{n}{m} \rceil, \\ 0 & \text{for} \quad \lceil \frac{m+n}{m} l + \frac{n}{m} \rceil \leq i < \lceil \frac{m+n}{m}(l+1) \rceil \end{cases} \tag{7.7}$$

if $m \neq 0$, and $\delta_{m,n}(\pi^i_{m,n}(b_1)) = 1$ for all i if $m = 0$.

Proof. Since $\delta_{m,n}(\pi^i_{m,n}(b_1)) = 1 \Leftrightarrow (m+n)l + 1 \leq 1 + mi \leq (m+n)l + n$, we have the lemma. □

Corollary 7.3.2. *If $m > n$, then $\delta_{m,n}(s) = 1$ implies $\delta_{m,n}(\pi_{m,n}(s)) = 0$. If $m < n$, then $\delta_{m,n}(s) = 0$ implies $\delta_{m,n}(\pi_{m,n}(s)) = 1$.*

Lemma 7.3.3. *Let (m, n) and (d, c) be pairs of coprime non-negative integers with $d/(c + d) < 1/2 < m/(m + n)$. Then we have*

(1) $\Omega_\infty(\mathcal{B}_{m,n}, \mathcal{B}_{d,c}) = \emptyset$,
(2) $\Omega_{\text{fin}}(\mathcal{B}_{m,n}, \mathcal{B}_{d,c}) = \{\omega_{\mathfrak{t},\mathfrak{b}} \mid (\mathfrak{t}, \mathfrak{b}) \in \mathrm{Top}(\mathcal{B}_{m,n}) \times \mathrm{Bot}(\mathcal{B}_{d,c})\}$.

Proof. (1) Obvious. (2) Let $\delta_1 = \delta_{m,n}$ and $\delta_2 = \delta_{d,c}$ and set $\pi_1 = \pi_{m,n}$ and $\pi_2 = \pi_{d,c}$. Let ω be any element of $\Omega_{\text{fin}}(\mathcal{B}_{m,n}, \mathcal{B}_{d,c})$. Write $\omega = \{(\pi^i_1(s_1), \pi^i_2(s_2)) \mid 1 \leq i \leq \ell\}$ as in (7.1). By definition we have $\delta_1(s_1) = 1$ and $\delta_2(s_2) = 0$. Then Cor. 7.3.2 says that $\delta_1(\pi_1(s_1)) = 0$ and $\delta_2(\pi_2(s_2)) = 1$. Thus $\ell = 1$ has to hold, namely $\omega = \omega_{\mathfrak{t},\mathfrak{b}}$ with $\mathfrak{t} = \pi_1(s_1)$ and $\mathfrak{b} = \pi_2(s_2)$. □

7.4. **Remarks on endomorphisms of an F-zip.** Let k be an algebraically closed field. Let Z be an F-zip over k and let $\mathcal{B} = (B, \delta)$ be the final type related to Z.

Lemma 7.4.1. *Let $\omega \in \Omega(\mathcal{B}, \mathcal{B})$. Assume $(b, b) \in \omega$ for a certain $b \in B$. Then ω is an infinite slice.*

Proof. Let b be an element of B such that $(b, b) \in \omega$. Then it is clear from the definition of slices that $(\pi^i(b), \pi^i(b)) \in \omega$ for all $i = 1, 2, \ldots$. This means $\omega \in \Omega_\infty(\mathcal{B}, \mathcal{B})$. □

Lemma 7.4.2. *Let $\omega_1, \ldots, \omega_n \in \Omega(\mathcal{B}, \mathcal{B})$ and let a_i be a non-zero element of $\mathbb{K}_{\omega_i}(k)$ for $i = 1, \ldots, n$. We denote by f_i the endomorphism $Z \to Z$ defined by (ω_i, a_i). Let ω be a slice defining $f_1 \circ \cdots \circ f_n$ (see Def. 7.1.3). If $\omega \in \Omega_\infty(\mathcal{B}, \mathcal{B})$, then $\omega_i \in \Omega_\infty(\mathcal{B}, \mathcal{B})$ for all $1 \leq i \leq n$.*

Proof. Clearly $\mathbb{K}_\omega$ contains $\mathbb{K}_{\omega_1} \cdots \mathbb{K}_{\omega_n}$ ($\subset \mathbb{G}_a$); hence we have $\mathbb{K}_\omega \supset \mathbb{K}_{\omega_i}$. If ω is an infinite slice, then $\mathbb{K}_\omega$ is finite; hence $\mathbb{K}_{\omega_i}$ is finite. This means that ω_i is an infinite slice. □

7.5. **A self-dual complex of F-zips.** Let

$$Z = (N, C, D, \varphi, \dot\varphi) \quad \text{and} \quad Z_1 = (N_1, C_1, D_1, \varphi_1, \dot\varphi_1)$$

be F-zips. Let $\mu : Z \to Z_1$ be a homomorphism of F-zips. Write $\mu_N : N \to N_1$ and let $\mu_C : C \to C_1$ and $\mu_D : D \to D_1$ be the restrictions of μ_N to C and D respectively.

Definition 7.5.1. (1) A homomorphism $\mu : Z \to Z_1$ is called *strictly surjective* if μ_N and μ_C are surjective.
(2) A homomorphism $\mu : Z_1 \to Z$ is called *strictly injective* if the dual $\mu^\vee : Z^\vee \to Z_1^\vee$ is strictly surjective.

Remark 7.5.2. Note that the surjectivity of μ_N implies that μ_D and $\mu_C^{(p)}$ are surjective.

Lemma 7.5.3. *Let $\mu : Z \to Z_1$ be a homomorphism of F-zips over S. The set of points of S where μ is strictly surjective (resp. strictly injective) is an open subset of S.*

Proof. It is enough to show the "strictly surjective" case. It suffices to show the case that S is affine. Apply [19], Th. 4.10 (i) to the cokernels of μ_N and μ_C. $\square$

For a strictly surjective homomorphism $\mu : Z \to Z_1$, we set $N_2 = \mathrm{Ker}(\mu : N \to N_1)$ with $C_2 = \mathrm{Ker}(\mu : C \to C_1)$ and $D_2 = \mathrm{Ker}(\mu : D \to D_1)$. Then since N_1 and C_1 are locally free, there exist isomorphisms $\varphi_2 : (N_2/C_2)^{(p)} \to D_2$ and $\dot\varphi : C_2^{(p)} \to N_2/D_2$ commuting diagrams of $\mathcal{O}_S$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N_2/C_2)^{(p)} & \longrightarrow & (N/C)^{(p)} & \longrightarrow & (N_1/C_1)^{(p)} & \longrightarrow & 0 \\ & & {\scriptstyle\varphi_2}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\varphi}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\varphi_1}\big\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & D_2 & \longrightarrow & D & \longrightarrow & D_1 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2^{(p)} & \longrightarrow & C^{(p)} & \longrightarrow & C_1^{(p)} & \longrightarrow & 0 \\ & & {\scriptstyle\dot\varphi_2}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\dot\varphi}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\dot\varphi_1}\big\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & N_2/D_2 & \longrightarrow & N/D & \longrightarrow & N_1/D_1 & \longrightarrow & 0, \end{array}$$

where the all horizontal complexes are exact. Thus we have an F-zip $Z_2 = (N_2, C_2, D_2, \varphi_2, \dot\varphi_2)$, which is called the *kernel* of μ, denoted by $\mathrm{Ker}(\mu)$. (If μ is not strictly surjective, we may not get an F-zip "$\mathrm{Ker}(\mu)$".) Similarly for a strictly injective homomorphism ν, we have its cokernel $\mathrm{Coker}(\nu) := \mathrm{Ker}(\nu^\vee)^\vee$.

Definition 7.5.4. Let Z be a polarized F-zip and Z_1 be an F-zip. A sequence of homomorphisms of F-zips of the form

$$\mathcal{C}^\bullet : \quad 0 \longrightarrow Z_1^\vee \xrightarrow{f^\vee} Z \xrightarrow{f} Z_1 \longrightarrow 0$$

is called a *self-dual complex* if

(1) $f \circ f^\vee : N_1^\vee \to N_1$ is zero,
(2) f is strictly surjective.

For a self-dual complex $\mathcal{C}^\bullet$ as above, we can define the *first cohomology* $H^1(\mathcal{C}^\bullet)$ by $\mathrm{Coker}(f^\vee : Z_1^\vee \to \mathrm{Ker}(f))$. One can check that $H^1(\mathcal{C}^\bullet)$ is a polarized F-zip.

7.6. **Constructing a non-trivial family of self-dual complexes of F-zips.** Let k be an algebraically closed field of characteristic p. Let

$$Z_1 = (N_1, C_1, D_1, \varphi_1, \dot\varphi_1)$$

be an F-zip over k. Let $\mathcal{B}_1$ be the final type of Z_1 (cf. §4.1). Assume

$$\Omega(\mathcal{B}_1^\vee, \mathcal{B}_1) = \{\omega_{\mathfrak{t},\mathfrak{b}} \mid \mathfrak{t} \in \mathrm{Top}(\mathcal{B}_1^\vee), \mathfrak{b} \in \mathrm{Bot}(\mathcal{B}_1)\}; \tag{7.8}$$

for example $Z_1 = \mathrm{fz}(H_{d,c}[p]_k)$ with $c > d$, see Lem. 7.3.3.

Proposition 7.6.1. *Let Z be a polarized F-zip $(N, C, D, \varphi, \dot{\varphi}, \langle\ ,\ \rangle)$ with self-dual complex of F-zips over k:*

$$\mathcal{C}_0^\bullet: \quad 0 \longrightarrow Z_1^\vee \xrightarrow{f_0^\vee} Z \xrightarrow{f_0} Z_1 \longrightarrow 0.$$

Assume $\mathcal{C}_0^\bullet$ has no splitting. Then there exists a self-dual complex

$$\mathcal{C}^\bullet: \quad 0 \longrightarrow Z_{1,S}^\vee \xrightarrow{f^\vee} Z_S \xrightarrow{f} Z_{1,S} \longrightarrow 0$$

over S smooth of finite type over k of relative dimension ≥ 1 with a section $\operatorname{Spec} k \to S$ *such that*

(1) $\mathcal{C}^\bullet \otimes k \simeq \mathcal{C}_0^\bullet$.
(2) $\mathcal{C}^\bullet$ *is "non-trivial", i.e., $f^\vee \neq \kappa \circ f_{0,S}^\vee$ for any automorphism κ of the polarized F-zip Z_S.*

Proof. Let $\mathcal{B}_* = (B_*, \delta_*)$ be the symmetric final type of Z_* and set $\pi_* = \pi_{\delta_*}$ for $* = \emptyset, 1$. Let $\{\omega_i\}$ be the set of the slices defining f_0 and let a_i be the string of f_0 at ω_i (see Def. 7.1.3). By the assumption that $\mathcal{C}_0^\bullet$ has no splitting, we have $\omega_i \in \Omega_{\text{fin}}(\mathcal{B}, \mathcal{B}_1)$. Note that $\Phi^{-1}(f_0^\vee)$ is given by $(a_i) \in \bigoplus_i \mathbb{K}_{\omega_i^\vee}(k)$.

Write $\omega_i = \{(\pi^v(b_i), \pi_1^v(c_i)) \mid 1 \leq v \leq l_i\}$ and put $\mathfrak{s}_i = \pi(b_i)$ and $\mathfrak{e}_i = \pi^{l_i}(b_i)$. Let pr denote the projection $B \times B_1 \to B$ and $\mathrm{pr}^\vee$ denote the projection $B_1^\vee \times B \to B$. First we prove

Claim 1. Every element of $\mathrm{pr}^\vee(\omega_j^\vee) \cap \mathrm{pr}(\omega_i)$ is of the form: $\mathfrak{e}_j^\vee = \mathfrak{s}_i$ or $\mathfrak{s}_j^\vee = \mathfrak{e}_i$.

Proof of Claim 1: By the assumption (7.8), the composition $Z_1^\vee \to Z \to Z_1$ constructed by $a_j \in \mathbb{K}_{\omega_j^\vee}$ and $a_i \in \mathbb{K}_{\omega_i}$ has to be defined by slices of the form $\omega_{\mathfrak{t},\mathfrak{b}}$ for some $(\mathfrak{t}, \mathfrak{b}) \in \mathrm{Top}(\mathcal{B}_1^\vee) \times \mathrm{Bot}(\mathcal{B}_1)$. Then any element of $\mathrm{pr}^\vee(\omega_j^\vee) \cap \mathrm{pr}(\omega_i)$ should be of the form: $\mathfrak{e}_j^\vee = \mathfrak{s}_i$ or $\mathfrak{s}_j^\vee = \mathfrak{e}_i$.

We say $\omega_i \sim \omega_j$ if $\mathrm{pr}(\omega_i) \cap \mathrm{pr}(\omega_j) \neq \emptyset$. Write $U = \{\omega_i\}/\sim$. Let $[\omega_i]$ denote the class of ω_i, i.e., $[\omega_i] = \{\omega_j | \omega_j \sim \omega_i\}$. For $u \in U$, we define a subset of B by

$$B_u = \bigcup_{\omega_i \in u} \mathrm{pr}(\omega_i);$$

then we can write $B_u = \{s_u, \pi(s_u), \cdots, \pi^{d(u)}(s_u)\}$ for a certain $s_u \in B$ and $d(u) \in \mathbb{Z}_{\geq 0}$; we put $e_u := \pi^{d(u)}(s_u)$; for any $\omega_i \in u$, we define d_i by

$$\pi^{d_i}(s_u) = \mathfrak{s}_i \qquad (0 \leq d_i \leq d(u)). \tag{7.9}$$

Let $P(a, b)$ be the property

$$\exists \omega_i \in [\omega_a], \exists \omega_j \in [\omega_b], \mathfrak{e}_i \in \mathrm{pr}^\vee(\omega_j^\vee) \cap \mathrm{pr}(\omega_i).$$

Set $U_+ = \{[\omega_a] \mid \exists b, P(a, b)\}$ and $U_- = \{[\omega_b] \mid \exists a, P(a, b)\}$. Since for all $a \in U_+$ there exists a unique b such that $P(a, b)$ holds, and for all $b \in U_-$ there exists a unique a such that $P(a, b)$ holds, we have the bijection

$$\gamma: \quad U_+ \xrightarrow{\sim} U_- \tag{7.10}$$

sending $[\omega_a]$ to $[\omega_b]$ satisfying $P(a, b)$.

Let $u \in U_+$. If $u \neq \gamma(u)$ we have

$$B_u \cap B_{\gamma(u)}^\vee = \{e_u = s_{\gamma(u)}^\vee\}, \qquad B_{\gamma(u)} \cap B_u^\vee = \{e_u^\vee = s_{\gamma(u)}\},$$

and otherwise

$$B_u \cap B_{\gamma(u)}^\vee = \{e_u = s_{\gamma(u)}^\vee, e_u^\vee = s_{\gamma(u)}\}.$$

Moreover for any $(v, v') \in U \times U$ we have $B_v \cap B^\vee_{v'} = \emptyset$ if $(v, v') \neq (u, \gamma(u)), (\gamma(u), u)$ for any $u \in U_+$.

Consider the parameter space $k[t_u | u \in U]$. Write $t = (t_u)$ and let f_t be the homomorphism $Z \to Z_1$ obtained by $\left(t^{p^{d_i}}_{[\omega_i]} a_i\right) \in \bigoplus_i \mathbb{K}_{\omega_i}$, see (7.9) for the definition of d_i. By the assumption (7.8), $f_t \circ f^\vee_t$ is given by strings $c_{\mathfrak{t},\mathfrak{b}}(t)$ at $\omega_{\mathfrak{t},\mathfrak{b}}$'s. Thus $f_t \circ f^\vee_t = 0$ if and only if

$$c_{\mathfrak{t},\mathfrak{b}}(t) = 0 \quad \text{for all } \mathfrak{t}, \mathfrak{b}. \tag{7.11}$$

Claim 2. The equations (7.11) in t are linear in $\{t_u^{p^{d(u)}} t_{\gamma(u)}\}_{u \in U_+}$ without any constant term.

Proof of Claim 2: For $v \in U$, let $f_{t,v}$ denote the $\left(\bigoplus_{\omega_i \in v} \mathbb{K}_{\omega_i}\right)$-part of f_t; then we can write $f_t = \sum f_{t,v}$. Note that $c_{\mathfrak{t},\mathfrak{b}}(t)$ is the sum of $\omega_{\mathfrak{t},\mathfrak{b}}$-coefficients of $f_{t,v} \circ f^\vee_{t,v'}$ for (i) $(v, v') = (u, \gamma(u))$ and (ii) $(v, v') = (\gamma(u), u)$ with $u \in U_+$. The both contributions of $f_{t,v} \circ f^\vee_{t,v'}$ at (i) $e_u = s^\vee_{\gamma(u)}$ and at (ii) $e^\vee_u = s_{\gamma(u)}$ are of the same form: const $\cdot\, t_u^{p^{d(u)}} t_{\gamma(u)}$. Thus we have Claim 2.

Let x be a new parameter. Put $\mathcal{R} = k[x, 1/x]$ if $U_+ \neq \emptyset$ and $\mathcal{R} = k$ if $U_+ = \emptyset$. Since $t_u = 1$ is a solution of (7.11), any solution of $\{t_u^{p^{d(u)}} t_{\gamma(u)} = x\}_{u \in U_+}$ gives a solution of (7.11) by Claim 2. We put

$$S' := \operatorname{Spec} \mathcal{R}[t_u \mid u \in U]/(t_u^{p^{d(u)}} t_{\gamma(u)} = x \mid u \in U_+)$$

and take as S the open part of S' where f_t is strictly surjective (see Lem. 7.5.3). Of course the required section $\operatorname{Spec} k \to S$ is defined by sending x and t_u to 1.

It remains to show that S is smooth over k of relative dimension ≥ 1. It suffices to show S' is smooth over $k[x, 1/x]$ in the case that $U_+ \neq \emptyset$. We can decompose U_+ as

$$U_+ = \bigsqcup_l \{u_l, \gamma(u_l), \ldots, \gamma^{n_l - 1}(u_l)\}$$

such that **(A)** $u_l \notin U_-$ and $\gamma^{n_l}(u_l) \notin U_+$ or **(B)** $\gamma^{n_l}(u_l) = u_l$. Since a fiber product of smooth morphisms is smooth (cf. [8], 17.3.3), it suffices to consider the simultaneous equations $t_u^{p^{d(u)}} t_{\gamma(u)} = x$ for $u \in \{u_l, \gamma(u_l), \ldots, \gamma^{n_l}(u_l)\}$ for each l. Note that $t_{\gamma^i(u_l)}$ for $i \geq 1$ is uniquely determined by x and t_{u_l}. Case **(A)**: We have no equation in t_{u_l}. Case **(B)**: Put $r_i = \sum_{j=i}^{n_l - 1} d(\gamma^i(u_l))$ for $0 \leq i < n_l$ with $r_{n_l} = 0$. We have a unique equation in t_{u_l}:

$$t_{u_l}^{p^{r_0} - (-1)^{n_l}} = x^{\sum_{i=1}^{n_l} (-1)^i p^{r_i}}.$$

This is an étale equation outside $x = 0$.

Finally let us show that $\mathcal{C}^\bullet$ satisfies the property (2). First note that the set of slices defining $f^\vee$ is the same as the set of slices defining $f^\vee_0$. Assume an element κ of $\operatorname{Aut}(Z_S)$ satisfied $f^\vee = \kappa \circ f^\vee_0$. Let $\mathfrak{B} = \{b \in \operatorname{Bot}(\mathcal{B}) \mid \exists \omega_i, \exists b' \in \operatorname{Bot}(\mathcal{B}^\vee_1), (b', b) \in \omega^\vee_i\}$. It follows from the construction of f that for any $b \in \mathfrak{B}$ there exists a "moving" slice ω' defining κ such that $\exists b_+ \in \mathfrak{B}$, $(b_+, b) \in \omega'$, where we say ω' is *moving* if the image of the string $\operatorname{Spec}(S) \to \mathbb{G}_a$ of κ at ω' (Def. 7.1.3) is positive dimensional. In this case we write $\omega' : b_+ \to b$. Then there exists at least one "cycle":

$$b_0 = b_N \xrightarrow{\omega'_{N-1}} \cdots \longrightarrow b_2 \xrightarrow{\omega'_1} b_1 \xrightarrow{\omega'_0} b_0,$$

where b_i are some elements of $\mathfrak{B}$ for $0 \leq i < N$ and ω'_i are some moving slices defining κ with $(b_{i+1}, b_i) \in \omega'_i$ for $0 \leq i < N$. Then by Lem. 7.4.1 and 7.4.2, ω'_i has to be an infinite slice. On the other hand if ω'_i is moving, then ω'_i has to be a finite slice. This is a contradiction. $\square$

8. A lifting to a self-dual complex of displays

The purpose of this section is to prove Prop. 8.3.1, where we construct a lifting of a family of self-dual complexes of F-zips (e.g., $\mathcal{C}$ constructed in Prop. 7.6.1) to a family of self-dual complexes of displays. For the construction we need to solve some equations in Witt vectors. Hence we start with preparing some lemmas to solve such equations.

8.1. **Lemmas.** Let Λ be a commutative ring of characteristic p.

Lemma 8.1.1. *Let $\Lambda' = \Lambda[x_0, \dots, x_n]$. Write $x = (x_0, \dots, x_n) \in W_n(\Lambda')$. For $a = (a_0, \dots, a_n)$ and $b \in W_n(\Lambda)$ and for $c \in \mathbb{Z}_{\geq 0}$, the equation $a \cdot {}^{\sigma^c} x - x = b$ in $W_n(\Lambda')$ is described as simultaneous equations in Λ' of the form*

$$a_0^{p^i} x_i^{p^c} - x_i = P_i(x_0, \dots, x_{i-1}) \qquad (0 \leq i \leq n)$$

for some $P_i \in \Lambda[x_0, \dots, x_{i-1}]$.

Proof. Let $\mathcal{R} = \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ be the ring of polynomials in $2(n+1)$ variables. Let $X = (X_0, \dots, X_n) \in W_n(\mathcal{R})$ and $Y = (Y_0, \dots, Y_n) \in W_n(\mathcal{R})$. Since the i-th entry of $X + Y \in W_n(\mathcal{R})$ is written as $\sigma_i(X_0, \dots, X_i; Y_0, \dots, Y_i)$ for some polynomial σ_i with coefficients in $\mathbb{Z}$, we have

$$X_0^{p^i} + \cdots + p^i X_i + Y_0^{p^i} + \cdots + p^i Y_i = \sigma_0(X_0; Y_0)^{p^i} + \cdots + p^i \sigma_i(X_0, \dots, X_i; Y_0, \dots, Y_i).$$

Hence $\sigma_i(X_0, \dots, X_i; Y_0, \dots, Y_i)$ has to be of the form

$$X_i + Y_i + Q_i(X_0, \dots, X_{i-1}, Y_0, \dots, Y_{i-1})$$

for a certain polynomial Q_i with coefficients in $\mathbb{Z}$.

The i-th entry of XY is written as $\pi_i(X_0, \dots, X_i; Y_0, \dots, Y_i)$ for some polynomial π_i with coefficients in $\mathbb{Z}$. We have

$$(X_0^{p^i} + \cdots + p^i X_i)(Y_0^{p^i} + \cdots + p^i Y_i) = \pi_0(X_0; Y_0)^{p^i} + \cdots + p^i \pi_i(X_0, \dots, X_i; Y_0, \dots, Y_i).$$

Since the characteristic of Λ is p, the x_i-coefficient of $\pi_i(x_0, \dots, x_i; y_0, \dots, y_i)$ is $y_0^{p^i}$ for the elements $(x_0, \dots, x_n)$ and $(y_0, \dots, y_n)$ of $W_n(\Lambda')$. $\square$

Lemma 8.1.2. *Let Γ be a finite set. Let $\gamma : \Gamma \to \Gamma$ be a map. Let $c_i \in \mathbb{Z}_{>0}$ and $a^{(i)}, b^{(i)} \in W_n(\Lambda)$ for $i \in \Gamma$. There exists a finite Λ-algebra R such that $\operatorname{Spec}(R) \to \operatorname{Spec}(\Lambda)$ is surjective and there exists a solution $(x^{(i)})$ ($i \in \Gamma$, $x^{(i)} \in W_n(R)$) of the simultaneous equations*

$$a^{(i)} \cdot {}^{\sigma^{c_i}} x^{(\gamma(i))} - x^{(i)} = b^{(i)}. \tag{8.1}$$

Proof. Let $\Gamma' = \bigcap_{r \in \mathbb{N}} \operatorname{Im} \gamma^r$. Then γ induces a bijective map $\gamma : \Gamma' \to \Gamma'$. Then Γ' is divided into γ-cycles. Let J be a γ-cycle in Γ'. First we solve the equations (8.1) only for $i \in J$. Let $j_0 \in J$ and set $j_r = \gamma^r(j_0)$. Write $\xi_r = x^{(j_r)}$ and put $\alpha_r = a^{(j_r)}$ and $\beta_r = b^{(j_r)}$ and $\sigma_r = \sigma^{c_{j_r}}$. Then our equations are written as

$$\alpha_r \cdot {}^{\sigma_r} \xi_{r+1} - \xi_r = \beta_r. \tag{8.2}$$

For $0 \leq r \leq |J|$ we put

$$\rho_r = \prod_{l=0}^{r-1} \sigma_l \quad \text{and} \quad A_r = \prod_{l=0}^{r-1} {}^{\rho_l} \alpha_l \tag{8.3}$$

with $\rho_0 = 1$ and $A_0 = 1$, and for $0 \le r < |J|$ we set

$$B_r = A_r \cdot {}^{\rho_r}\beta_r. \tag{8.4}$$

Then we have $A_{r+1} \cdot {}^{\rho_{r+1}}\xi_{r+1} - A_r \cdot {}^{\rho_r}\xi_r = B_r$; hence

$$A_{|J|} \cdot {}^{\rho_{|J|}}\xi_0 - \xi_0 = \sum_{0 \le r < |J|} B_r. \tag{8.5}$$

By Lem. 8.1.1, there is a finite Λ-algebra R' such that $\mathrm{Spec}(R') \to \mathrm{Spec}(\Lambda)$ is surjective and we have a solution $\xi_0 \in W_n(R')$ of (8.5). From (8.2) we can find a finite Λ-algebra R'' with surjective $\mathrm{Spec}(R'') \to \mathrm{Spec}(\Lambda)$ such that the remaining ξ_i are in $W_n(R'')$. Doing the same thing for the other γ-cycles in Γ' successively, we get a finite Λ-algebra R with surjective $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$ such that we have a solution $(x^{(i)})$ of the equations (8.1) for $i \in \Gamma'$.

For $i \in \Gamma \setminus \Gamma'$, there is a unique sequence $(i, \gamma(i), \dots, \gamma^l(i))$ satisfying $\gamma^l(i) \in \Gamma'$ and $\gamma^r(i) \notin \Gamma'$ for $r < l$. By the descending induction on r, we obtain a solution $x^{(\gamma^r(i))}$ of (8.1). □

Remark 8.1.3. Lem. 8.1.2 holds also for $c_i \in \mathbb{Z}_{\ge 0}$ if for every γ-cycle J in Γ satisfying $\rho_{|J|} = \mathrm{id}$, there exists a solution of (8.5): $(A_{|J|} - 1)\xi_0 = \sum_{0 \le r < |J|} B_r$.

Let $W_{\mathbb{Q}}(\Lambda) = W(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that $W_{\mathbb{Q}}(\Lambda) = \varinjlim_n W(\Lambda) \otimes_{\mathbb{Z}_p} (1/p^n)\mathbb{Z}_p$.

Corollary 8.1.4. *Assume Λ is of finite type over a perfect field k. Let n be a non-negative integer. Let Γ be a finite set with a map $\gamma : \Gamma \to \Gamma$. Let $c_i \in \mathbb{Z}_{>0}$ and $a^{(i)}, b^{(i)} \in W_{\mathbb{Q}}(\Lambda)$ for $i \in \Gamma$. There exists a finite Λ-algebra R' such that* $\mathrm{Spec}(R') \to \mathrm{Spec}(\Lambda)$ *is surjective and there exists a solution $(x^{(i)})$ ($i \in \Gamma$, $x^{(i)} \in W_{\mathbb{Q}}(R')/I_{R',n}$) of the simultaneous equations*

$$a^{(i)} \cdot {}^{\sigma^{c_i}}x^{(\gamma(i))} - x^{(i)} \equiv b^{(i)} \qquad (\mathrm{mod}\, I_{R',n}). \tag{8.6}$$

Proof. Let m be a non-negative integer such that $a^{(i)}, b^{(i)} \in W(\Lambda) \otimes_{\mathbb{Z}_p} (1/p^m)\mathbb{Z}_p$ for all $i \in \Gamma$. Let R be the finite Λ-algebra obtained in Lem. 8.1.2 for $p^m a^{(i)}, p^m b^{(i)}$ modulo $I_{R,m+n}$; then there exist $y^{(i)} \in W(R)$ for $i \in \Gamma$ such that

$$p^n a^{(i)} \cdot {}^{\sigma^{c_i}}y^{(\gamma(i))} - y^{(i)} \equiv p^n b^{(i)} \qquad (\mathrm{mod}\, I_{R,m+n}). \tag{8.7}$$

Note that R is of finite type over k. There exists a finite R-algebra R' such that $(R')^{p^m} = R$ and $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective. Then we have $I_{R,m+n} = p^m I_{R',n}$; hence $(x^{(i)}) = (p^{-m}y^{(i)})$ is a solution of (8.6). □

Remark 8.1.5. Cor. 8.1.4 holds even for $c_i \in \mathbb{Z}_{\ge 0}$ if there exists a finite Λ-algebra R'' with surjective $\mathrm{Spec}(R'') \to \mathrm{Spec}\,\Lambda$ such that there is a solution of (8.6) for $i \in \Gamma' = \bigcap_{r \in \mathbb{N}} \mathrm{Im}\,\gamma^r$. See the last paragraph in the proof of Lem. 8.1.2.

8.2. **Minimal displays.** Let ξ be a Newton polygon without the étale segment $(0,1)$. We denote by $M(\xi)$ the display over $\mathbb{F}_p$ of the minimal p-divisible group $H(\xi)$ (§5.1). Write $M(\xi) = (P(\xi), Q(\xi), \mathcal{F}, \mathcal{V}^{-1})$. Remark that $P(\xi)$ here is canonically identified with that at (5.3).

For later use, we need to describe $M(\xi)$ explicitly for the cases $\xi = (d,c)$ and (c,d) for $\gcd(c,d) = 1$ and $c > d > 0$. We write $M_{c,d} = M((c,d))$ and $P_{c,d} = P((c,d))$, etc. First we introduce a "good" basis of $P_{c,d}$ and a normal decomposition $P_{c,d} = L_{c,d} \oplus T_{c,d}$, which defines $Q_{c,d} = L_{c,d} \oplus I_{\mathbb{F}_p}T_{c,d}$. Let $\{e_0, \dots, e_{c+d-1}\}$ be a minimal basis of $P_{c,d}$ (see § 5.1). Let $\alpha(e_i)$ denote the largest integer α such that

$i + \alpha d < c + d$, namely $\alpha(e_i) = \lfloor (c+d-i)/d \rfloor$. Note that $\alpha(e_i) \geq 1$ for all $i < c$. We set $x_0 = e_0$ and define inductively x_i ($i \in \mathbb{N}$) by

$$x_{i+1} = \mathcal{V}^{-1}\mathcal{F}^{\alpha_i} x_i \qquad \text{with} \quad \alpha_i := \alpha(x_i).$$

Note that $x_{i+d} = x_i$ and $\{x_i \mid i \in \mathbb{Z}/d\mathbb{Z}\} = \{e_0, \dots, e_{d-1}\}$; then we have $|\alpha_i - \alpha_j| \leq 1$ for all $i, j \in \mathbb{Z}/d\mathbb{Z}$. Clearly $M_{c,d}$ is given by

$$P_{c,d} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \bigoplus_{s=0}^{\alpha_i} \mathbb{Z}_p \mathcal{F}^s x_i$$

with normal decomposition $P_{c,d} = L_{c,d} \oplus T_{c,d}$ defined by

$$L_{c,d} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathbb{Z}_p \mathcal{F}^{\alpha_i} x_i \qquad \text{and} \qquad T_{c,d} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \bigoplus_{s=0}^{\alpha_i - 1} \mathbb{Z}_p \mathcal{F}^s x_i.$$

Similarly $M_{d,c}$ is given by

$$P_{d,c} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \bigoplus_{s=0}^{\alpha_i} \mathbb{Z}_p \mathcal{V}^{-s} y_i$$

with normal decomposition $P_{d,c} = L_{d,c} \oplus T_{d,c}$ defined by

$$L_{d,c} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \bigoplus_{s=0}^{\alpha_i - 1} \mathbb{Z}_p \mathcal{V}^{-s} y_i \qquad \text{and} \qquad T_{d,c} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathbb{Z}_p \mathcal{V}^{-\alpha_i} y_i.$$

We define an alternating bilinear form $(\ ,\ )$ on $P_{c,d} \oplus P_{d,c}$ by $(P_{c,d}, P_{c,d}) = 0$ and $(P_{d,c}, P_{d,c}) = 0$, and

$$(\mathcal{F}^k x_i, \mathcal{V}^{-l} y_j) = \delta_{ij}\delta_{kl}.$$

Clearly $(\ ,\ )$ gives a principal quasi-polarization on $M_{c,d} \oplus M_{d,c}$.

8.3. **Construction of a lifting of a self-dual complex of F-zips.** Let $\xi = \sum_{l=1}^{\mathrm{t}} (m_l, n_l)$ be a symmetric Newton polygon with $\lambda_1 \leq \cdots \leq \lambda_{\mathrm{t}}$, where $\lambda_l = m_l/(m_l + n_l)$. Put $\xi' = \sum_{l=2}^{\mathrm{t}-1} (m_l, n_l)$ and set $(d, c) := (m_1, n_1)$. We assume $c > d > 0$. Let $M_{c,d} = (P_{c,d}, Q_{c,d}, \mathcal{F}, \mathcal{V}^{-1})$ and $M_{d,c} = (P_{d,c}, Q_{d,c}, \mathcal{F}, \mathcal{V}^{-1})$ be the minimal displays, which were explicitly described in the previous subsection; hence we will freely use the notation in §8.2. Let Λ be a commutative ring of finite type over a perfect field k. Put $M_1 = (M_{d,c})_\Lambda$ and set $Z_1 = M_1/I_\Lambda M_1$; then $Z_1^\vee = M_1^t/I_\Lambda M_1^t$ with $M_1^t = (M_{c,d})_\Lambda$. For any display $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ over Λ, let $^-$ denote the natural projection $P \to P/I_\Lambda P$. Let $Z = (N, C, D, \varphi, \dot{\varphi}, \langle\ ,\ \rangle)$ be a polarized F-zip over Λ and f be a strictly surjective homomorphism $Z \to Z_1$ making a self-dual complex

$$\mathcal{C}^\bullet : \quad 0 \longrightarrow Z_1^\vee \xrightarrow{f^\vee} Z \xrightarrow{f} Z_1 \longrightarrow 0. \tag{8.8}$$

The following is a key proposition in this paper, where for any lifting of $H^1(\mathcal{C}^\bullet)$ to a display we construct a lifting of $\mathcal{C}^\bullet$ to a self-dual complex of displays. The original idea of the construction is found in [17], §7.

Proposition 8.3.1. *Let M' be any principally quasi-polarized display over Λ with $M'/I_\Lambda M' \simeq H^1(\mathcal{C}^\bullet)$. Let $\langle\ ,\ \rangle'$ be a quasi-polarization on the minimal display $M(\xi')$. Assume we are given an isogeny*

$$(M(\xi'), \langle\ ,\ \rangle')_\Lambda \xrightarrow{\rho'} M' \tag{8.9}$$

as quasi-polarized displays. Then for a finite surjective morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$*, there exist a principally quasi-polarized display* $\mathcal{M}$ *over* R *with an isogeny of quasi-polarized displays*

$$(M(\xi), \langle\ ,\ \rangle)_R \xrightarrow{\ \rho\ } \mathcal{M} \tag{8.10}$$

for a certain polarization $\langle\ ,\ \rangle$ *on* $M(\xi)$ *and an isomorphism* $\kappa : \mathcal{M}/I_R\mathcal{M} \to Z_R$ *and a surjective homomorphism* $\phi : \mathcal{M} \to M_1$ *making a self-dual complex*

$$\mathcal{D}^\bullet : \quad 0 \longrightarrow M_{1,R}^t \xrightarrow{\ \phi^t\ } \mathcal{M} \xrightarrow{\ \phi\ } M_{1,R} \longrightarrow 0$$

such that

(1) $H^1(\mathcal{D}^\bullet) \simeq M'_R$,

(2) *we have a commutative diagram*

$$\begin{array}{ccc} M_{1,R}^t & \xrightarrow{\ \phi^t\ } & \mathcal{M} \\ \| & & \uparrow \rho \\ (M_{c,d})_R & \xrightarrow{\ \subset\ } & M(\xi)_R \end{array}$$

(3) *we have a commutative diagram*

$$\begin{array}{ccccccccc} \overline{\mathcal{D}}^\bullet : 0 & \longrightarrow & M_{1,R}^t/I_R M_{1,R}^t & \xrightarrow{\ \overline{\phi}^t\ } & \mathcal{M}/I_R\mathcal{M} & \xrightarrow{\ \overline{\phi}\ } & M_{1,R}/I_R M_{1,R} & \longrightarrow & 0 \\ & & \downarrow = & & \kappa \downarrow \simeq & & \downarrow = & & \\ \mathcal{C}_R^\bullet : 0 & \longrightarrow & Z_{1,R}^\vee & \xrightarrow{\ f^\vee\ } & Z_R & \xrightarrow{\ f\ } & Z_{1,R} & \longrightarrow & 0. \end{array}$$

Moreover, assume that with respect to a section $\mathrm{Spec}(k) \to \mathrm{Spec}(\Lambda)$

(i) *we can write* Z *and* M' *as* $Z_k \otimes \Lambda$ *and* $M'_k \otimes \Lambda$ *respectively,*

(ii) ρ' *is a trivial family,*

(iii) $\mathcal{C}^\bullet$ *is non-trivial (see Prop. 7.6.1, (2) for the definition);*

then ρ *is a non-trivial family.*

Proof. We are given a complex

$$\mathcal{C}^\bullet : \quad 0 \longrightarrow N_1^\vee \xrightarrow{\ f^\vee\ } N \xrightarrow{\ f\ } N_1 \longrightarrow 0 \tag{8.11}$$

and $H^1(\mathcal{C}^\bullet) \simeq N'$. A technical lemma (Lem. 8.3.2 below) shows that for a finite surjective morphism $\mathrm{Spec}(\Lambda') \to \mathrm{Spec}(\Lambda)$, there exists a lift $\overline{v}_{i,s} \in N_{\Lambda'}$ of $\overline{\mathcal{V}^{-s}y_i}$ ($i \in \mathbb{Z}/d\mathbb{Z}$, $0 \le s \le \alpha_i$) such that $\overline{v}_{i,s} \in C_{\Lambda'}$ ($s < \alpha_i$) and $\dot{\varphi}^{-1}(\overline{v}_{i,s+1}) = 1 \otimes \overline{v}_{i,s}$ for $0 \le s < \alpha_i$, and

$$\langle \overline{v}_{i,s}, \overline{v}_{i',s'} \rangle = 0 \tag{8.12}$$

for all $i, i' \in \mathbb{Z}/d\mathbb{Z}$ and for all $0 \le s \le \alpha_i$ and $0 \le s' \le \alpha_{i'}$. We replace Λ by Λ'. For any $\overline{z} \in N'$, let $\overline{u}(\overline{z})$ be an element of $\mathrm{Ker}\, f$ uniquely determined by $(\overline{u}(\overline{z}) \bmod N_1^\vee) = \overline{z}$ and

$$\langle \overline{u}(\overline{z}), \overline{v}_{i,s} \rangle = 0 \qquad \text{for} \quad \forall i \in \mathbb{Z}/d\mathbb{Z}, \quad 0 \le \forall s \le \alpha_i. \tag{8.13}$$

Thus we have generators of N:

$$\overline{v}_{i,s} \quad (1 \le i \le d, 0 \le s \le \alpha_i), \quad \overline{u}(\overline{z}) \quad (\overline{z} \in N'), \quad \overline{\mathcal{F}^s x_i} \quad (1 \le i \le d, 0 \le s \le \alpha_i).$$

We define $\overline{z}_i \in N'$ ($i \in \mathbb{Z}/d\mathbb{Z}$) by

$$\overline{z}_i = \varphi(1 \otimes \overline{v}_{i-1,\alpha_{i-1}}) - \overline{v}_{i,0} \mod N_1^\vee; \tag{8.14}$$

then we can write

$$\varphi(1\otimes\overline{v}_{i-1,\alpha_{i-1}})-\overline{v}_{i,0}=\overline{u}(\overline{z}_i)+\sum_{j\in\mathbb{Z}/d\mathbb{Z}}\sum_{s=0}^{\alpha_j}\overline{d}_{i,j,s}\overline{\mathcal{F}^s x_j},$$

for some $\overline{d}_{i,j,s}\in\Lambda$. By (8.12) and (8.13), we have $\overline{d}_{i,j,s}=\langle\overline{\varphi}(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),\overline{v}_{j,s}\rangle$. If $s>0$, then we have

$$\overline{d}_{i,j,s}=\langle(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),\dot{\varphi}^{-1}(\overline{v}_{j,s})\rangle^{(p)}=\langle(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),1\otimes\overline{v}_{j,s-1})\rangle^{(p)}=0.$$

Put $\overline{d}_{i,j}:=\overline{d}_{i,j,0}$, namely

$$\overline{d}_{i,j}:=\langle\varphi(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),\overline{v}_{j,0}\rangle. \tag{8.15}$$

Note that all relations involved with $\{\overline{v}_{i,s}\}$ are generated by $\dot{\varphi}^{-1}(\overline{v}_{i,s+1})=1\otimes\overline{v}_{i,s}$ for $0\le s<\alpha_i$ and the relations of the form

$$\varphi(1\otimes\overline{v}_{i-1,\alpha_{i-1}})-\overline{v}_{i,0}=\overline{u}(\overline{z}_i)+\sum_{j\in\mathbb{Z}/d\mathbb{Z}}\overline{d}_{i,j}\overline{x_j}. \tag{8.16}$$

For later use, we show

$$\overline{d}_{i,j}=\overline{d}_{j,i}-\langle\overline{z}_i,\overline{z}_j\rangle, \tag{8.17}$$

where the pairing on the second term is on N'. Indeed

$$\overline{d}_{i,j}=\langle\varphi(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),\overline{v}_{j,0}\rangle=\overline{d}_{j,i}-\langle\varphi(1\otimes\overline{v}_{i-1,\alpha_{i-1}}),\overline{u}(\overline{z}_j)\rangle. \tag{8.18}$$

By (8.14), (8.13) and the fact $\langle N_1^\vee,\overline{u}(\overline{z}_j)\rangle=0$, this is equal to $\overline{d}_{j,i}-\langle\overline{z}_i,\overline{u}(\overline{z}_j)\rangle$, which is also equal to $\overline{d}_{j,i}-\langle\overline{z}_i,\overline{z}_j\rangle$, since $\langle\overline{z}_i,N_1^\vee\rangle=0$.

Let R be a "sufficient large" Λ-algebra determined later. We define a projective $W_{\mathbb{Q}}(R)$-module

$$\mathbb{P}_R=P(\xi)_R\otimes\mathbb{Q}\qquad\text{with}\quad P(\xi)=P_{c,d}\oplus P(\xi')\oplus P_{d,c}.$$

Note that $\mathbb{P}_R$ is equipped with an alternating form $\langle\ ,\ \rangle$ induced by $(\ ,\ )$ on $P_{c,d}\oplus P_{d,c}$ and $\langle\ ,\ \rangle'$ on $P(\xi')$. We also have $W_{\mathbb{Q}}(R)$-linear homomorphisms $\mathcal{F}:\mathbb{P}_R^\sigma\to\mathbb{P}_R$ and $\mathcal{V}^{-1}:\mathbb{P}_R^\sigma\to\mathbb{P}_R$ with $\mathbb{P}_R^\sigma=W_{\mathbb{Q}}(R)\otimes_{\sigma,W_{\mathbb{Q}}(R)}\mathbb{P}_R$. Put

$$F(*):=\mathcal{F}(1\otimes *)\qquad\text{and}\qquad V^{-1}(*):=\mathcal{V}^{-1}(1\otimes *), \tag{8.19}$$

and for $s\in\mathbb{N}$ we inductively define $F^s(*)$ and $V^{-s}(*)$ by $F^s(*)=\mathcal{F}(1\otimes F^{s-1}(*))$ and $V^{-s}(*)=\mathcal{V}^{-1}(1\otimes V^{-(s-1)}(*))$ respectively. We write

$$\mathbb{P}_R=\bigoplus_{l=1}^{\mathfrak{t}}\mathbb{P}_R^{(l)}\qquad\text{with}\quad\mathbb{P}_R^{(l)}=P_{m_l,n_l,R}\otimes\mathbb{Q}$$

and set

$$\mathbb{P}'_R=\bigoplus_{l=2}^{\mathfrak{t}-1}\mathbb{P}_R^{(l)}.$$

For $2\le l\le\mathfrak{t}-1$, let $e_0^{(l)},\dots,e_{m_l+n_l-1}^{(l)}$ be a minimal basis of P_{m_l,n_l}. Write $M'=(P',Q',\mathcal{F},\mathcal{V}^{-1},\langle\ ,\ \rangle')$. Note that P' is in $\mathbb{P}_R$.

Let us define a principally quasi-polarized display $\mathcal{M}=(\mathcal{P},\mathcal{Q},\mathcal{F},\mathcal{V}^{-1},\langle\ ,\ \rangle)$. We will define $\mathcal{P}$ to be a submodule of $\mathbb{P}_R$ generated by $P_{c,d}$ and some elements

$$\begin{cases}V^{-s}v_i\in\mathbb{P}_R & (1\le i\le d\text{ and }0\le s\le\alpha_i),\\ u(z)\in\mathbb{P}_R^{(1)}\oplus\mathbb{P}'_R & (z\in P')\end{cases}$$

of the form $v_i = y_i + \sum_{l=2}^{\mathfrak{t}} A_i^{(l)}$ with $A_i^{(l)} \in \mathbb{P}_R^{(l)}$ and $u(z) = z + B(z)$ with $B(z) \in \mathbb{P}_R^{(\mathfrak{t})}$, where $A_i^{(l)}$ and $B(z)$ will be chosen later such that $\mathcal{M}$ has the required properties.

Let $z_i \in P'$ ($i \in \mathbb{Z}/d\mathbb{Z}$) be a lift of $\overline{z}_i$ defined in (8.14) and we write $z_i = \sum_{l=2}^{\mathfrak{t}-1} z_i^{(l)}$ with $z_i^{(l)} \in \mathbb{P}_R^{(l)}$. Put $v_i' = y_i + \sum_{l=2}^{\mathfrak{t}-1} A_i^{(l)}$. Write

$$A_i^{(l)} = \sum_{j=0}^{m_l+n_l-1} a_{ij}^{(l)} e_j^{(l)}, \qquad a_{ij}^{(l)} \in W_{\mathbb{Q}}(R) \qquad \text{for} \quad 2 \leq l < \mathfrak{t}.$$

We define $a_{ij}^{(l)}$ ($i \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq j < m_l + n_l$, $2 \leq l < \mathfrak{t}$) as satisfying

$$\begin{cases} FV^{-\alpha_{i-1}} v_{i-1}' - v_i' = z_i & \text{for } 1 \leq i < d, \\ FV^{-\alpha_{d-1}} v_{d-1}' - v_0' \equiv z_d \pmod{I_{R,\mathfrak{n}} P(\xi')_R} & \end{cases} \tag{8.20}$$

for a sufficient large $\mathfrak{n} \in \mathbb{N}$ (OK. if $\mathfrak{n} > \alpha_i$ for all $i \in \mathbb{Z}/d\mathbb{Z}$). Setting

$$U_{j,i} = (FV^{-\alpha_{j-1}}) \cdots (FV^{-\alpha_{i+1}})(FV^{-\alpha_i}) \qquad (0 \leq i < j \leq d),$$

we obtain the equation

$$U_{d,0} A_0^{(l)} - A_0^{(l)} \equiv \sum_{i=1}^{d} U_{d,i} z_i^{(l)} \pmod{I_{R,\mathfrak{n}} P_{m_l,n_l,R}}. \tag{8.21}$$

Comparing the $e_j^{(l)}$-coefficients of the both sides of (8.21) for each $0 \leq j < m_l + n_l$, we have simultaneous equations as in Cor. 8.1.4. Hence we can choose a finite Λ-algebra R with surjection $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$ such that there is a solution $\{a_{0j}^{(l)}\}$ ($2 \leq l \leq \mathfrak{t} - 1$) of (8.21). We define $a_{ij}^{(l)} \in W_{\mathbb{Q}}(R)$ for $i > 0$ by

$$v_i' := U_{i,0} v_0' - \sum_{j=1}^{i} U_{i,j} z_j.$$

Then v_i' ($i \in \mathbb{Z}/d\mathbb{Z}$) satisfy (8.20).

We determine $B(z)$ uniquely by the equations:

$$\langle u(z), V^{-s} v_i' \rangle = 0 \tag{8.22}$$

for $i \in \mathbb{Z}/d\mathbb{Z}$ and $0 \leq s \leq \alpha_i$.

For each $0 \leq i \leq j < d$, we choose a lift $d_{i,j} \in W(\Lambda)$ of $\overline{d}_{i,j}$ and for $0 \leq j < i < d$ we set

$$d_{i,j} = d_{j,i} - \langle z_i, z_j \rangle, \tag{8.23}$$

where the pairing of the second term is on P'. By (8.17) we see that $d_{i,j}$ are lifts of $\overline{d}_{i,j}$ even for $0 \leq j < i < d$.

We define $A_i^{(\mathfrak{t})}$ for every $i \in \mathbb{Z}/d\mathbb{Z}$ as satisfying the equations:

$$\begin{array}{llcll} (\mathbf{A}) & \langle v_i, V^{-s} v_j \rangle & \equiv & 0 \pmod{I_{R,\mathfrak{n}}} & \text{for} \quad i, j \in \mathbb{Z}/d\mathbb{Z},\ 0 \leq s \leq \alpha_j, \\ (\mathbf{B}) & \langle FV^{-\alpha_{i-1}} v_{i-1}, v_j \rangle & \equiv & d_{i,j} \pmod{I_{R,\mathfrak{n}}} & \text{for} \quad 0 \leq i \leq j < d. \end{array}$$

Before solving this collection of equations, we give some remarks. First from **(A)** and (3.5) we have

$$\langle V^{-s} v_i, V^{-s'} v_{i'} \rangle \equiv 0 \pmod{I_R} \tag{8.24}$$

for $i, i' \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq s \leq \alpha_i$ $0 \leq s' \leq \alpha_{i'}$. Secondly we claim that **(A)** and **(B)** imply

$$\langle FV^{-\alpha_{i-1}} v_{i-1}, v_j \rangle \equiv d_{i,j} \pmod{I_R} \qquad \text{for all} \quad i, j \in \mathbb{Z}/d\mathbb{Z}. \tag{8.25}$$

Indeed by **(A)** and (8.22) we have

$$FV^{-\alpha_{i-1}}v_{i-1} - v_i \equiv u(z_i) + \sum_{j\in\mathbb{Z}/d\mathbb{Z}} d'_{i,j}x_j \pmod{I_{R,\mathfrak{n}}P(\xi)}$$

with $d'_{i,j} := \langle FV^{-\alpha_{i-1}}v_{i-1}, v_j\rangle$. It suffices to show $d'_{i,j} \equiv d'_{j,i} - \langle z_i, z_j\rangle \pmod{I_R}$. By (3.5), (8.20) and (8.22) we have

$$\begin{aligned} d'_{i,j} &\equiv \langle FV^{-\alpha_{i-1}}v_{i-1}, FV^{-\alpha_{j-1}}v_{j-1} - u(z_j) - \sum_{k\in\mathbb{Z}/d\mathbb{Z}} d'_{j,k}x_k\rangle \pmod{I_{R,\mathfrak{n}}} \\ &\equiv d'_{j,i} - \langle FV^{-\alpha_{i-1}}v_{i-1}, u(z_j)\rangle \equiv d'_{j,i} - \langle z_i, z_j\rangle \pmod{I_R}. \end{aligned}$$

Write

$$A_i^{(\mathfrak{t})} = \sum_{j\in\mathbb{Z}/d\mathbb{Z}, 0\le s\le\alpha_j} \xi_{i,j,s}\mathcal{F}^s x_j.$$

Let us rewrite **(A)** and **(B)** by using $\{\xi_{i,j,s}\}$. Since $\mathcal{V}^s x_i = \mathcal{V}^{s-1}\mathcal{F}^{\alpha_{i-1}}x_{i-1} = p^{s-1}\mathcal{F}^{\alpha_{i-1}-s+1}x_{i-1}$ for $s\ge 1$, **(A)** is translated as

$$p^{e(s)}\cdot{}^{\sigma^s}\xi_{\gamma(i,j,s)} - \xi_{i,j,s} \equiv \beta_{i,j,s} \qquad \pmod{I_{R,\mathfrak{n}}}, \tag{8.26}$$

where $\gamma(i,j,s) = (j, i-1, \alpha_{i-1}-s+1)$ and $e(s) = 1-s$ for $s\ge 1$ and $\gamma(i,j,0) = (j,i,0)$ and $e(0) = 0$, and $\beta_{i,j,s}$ is a constant $\langle v'_i - y_i, {}^{\sigma^s}(v'_j - y_j)\rangle$. Note that $\beta_{i,j,0} + \beta_{j,i,0} = 0$.

Since $\mathcal{F}^{-1}\mathcal{V}^{\alpha_{i-1}}x_j$ is equal to $p^{\alpha_{i-1}-1}\mathcal{F}^{\alpha_{j-1}-\alpha_{i-1}}x_{j-1}$ if $\alpha_{i-1}\le\alpha_{j-1}$ and to $p^{\alpha_{i-1}-2}\mathcal{F}^{\alpha_{j-2}}x_{j-2}$ if $\alpha_{i-1} > \alpha_{j-1}$ (here we used $|\alpha_{j-1}-\alpha_{i-1}|\le 1$), **(B)** is translated as

$$p^{e'(j,i,0)}\cdot{}^{\sigma^{\alpha_{i-1}+1}}\xi_{\gamma'(j,i,0)} - \xi_{j,i,0} \equiv \beta'_{i,j} \qquad \pmod{I_{R,\mathfrak{n}}} \tag{8.27}$$

with a constant $\beta'_{i,j}$ (determined by $d_{i,j}$ and $A_{i'}^{(l)}$'s), where $\gamma'(j,i,0) = (i-1, j-1, \alpha_{j-1}-\alpha_{i-1})$ and $e'(j,i,0) = -(\alpha_{i-1}-1)$ for $\alpha_{i-1}\le\alpha_{j-1}$, and $\gamma'(j,i,0) = (i-1, j-2, \alpha_{j-2})$ and $e'(j,i,0) = -(\alpha_{i-1}-2)$ for $\alpha_{i-1} > \alpha_{j-1}$. By applying Cor. 8.1.4 to (8.26) for $0\le i,j<d$ and $s>0$ and for $0\le i<j<d$ and $s=0$ and (8.27) for $0\le i\le j<d$, we can choose R with finite surjective morphism $\mathrm{Spec}(R)\to\mathrm{Spec}(\Lambda)$ such that there exists a solution of the simultaneous equations. Now we have finished defining $\mathcal{P}$.

In order to define $\mathcal{Q}$, it suffices to define a normal decomposition $\mathcal{P} = L\oplus T$; then $\mathcal{Q} = L\oplus I_R T$. Let $P' = L'\oplus T'$ be a normal decomposition of P'. We define L to be the submodule of $\mathcal{P}$ generated by $L_{c,d}$, $u(z)$ $(z\in L')$, $V^{-s}v_i$ $(0\le s<\alpha_i)$ and T to be the submodule of $\mathcal{P}$ generated by $T_{c,d}$, $u(z)$ $(z\in T')$, $V^{-\alpha_i}v_i$. We have to show that $(\mathcal{P},\mathcal{Q},\mathcal{F},\mathcal{V}^{-1},\langle\ ,\ \rangle|_{\mathcal{P}})$ is a principally quasi-polarized display. It suffices to check that

(a) $\mathcal{V}^{-1}$ induces a well-defined surjective map $\mathcal{V}^{-1}:\mathcal{Q}^\sigma\to\mathcal{P}$ and
(b) $\langle\ ,\ \rangle|_{\mathcal{P}}$ is a perfect pairing on $\mathcal{P}$.

(b) follows immediately from (8.22) and (8.24). (a) In order to show that $\mathcal{V}^{-1}:\mathcal{Q}^\sigma\to\mathcal{P}$ is well-defined, it suffices to show that the elements $V^{-1}u(z)$ $(z\in L')$ and $V^{-1}({}^\tau 1\cdot u(z))$ $(z\in P')$ of $\mathbb{P}_R$ are in $\mathcal{P}$. For $V^{-1}u(z)$ $(z\in L')$, it is enough to show that

$$V^{-1}u(z) - u(V^{-1}z) \equiv 0 \pmod{P_{c,d,R}} \qquad \text{for}\quad z\in L'. \tag{8.28}$$

This is equivalent to $\langle V^{-1}u(z) - u(V^{-1}z), V^{-s}v'_j\rangle \in W(R)$. Since

$$\langle V^{-1}u(z) - u(V^{-1}z), V^{-s}v'_j\rangle = \langle V^{-1}u(z), V^{-s}v'_j\rangle,$$

it suffices to check that

$$\langle V^{-1}u(z), V^{-s}v'_j\rangle \in W(R). \tag{8.29}$$

For $s>0$, (8.29) follows from

$${}^{\tau}\langle V^{-1}u(z), V^{-s}v'_j\rangle = \langle u(z), V^{-s+1}v'_j\rangle = 0.$$

For $s=0$, from (8.20) we have

$$\langle V^{-1}u(z), V^{-s}v'_j\rangle \equiv \langle V^{-1}u(z), FV^{-\alpha_{j-1}}v'_{j-1}\rangle \qquad (\operatorname{mod} W(R)) \tag{8.30}$$

and the RHS of (8.30) is equal to ${}^{\sigma}\langle u(z), V^{-\alpha_{j-1}}v'_{j-1}\rangle = 0$. Hence (8.29) holds also for $s=0$. Similarly one can show that $V^{-1}({}^{\tau}1\cdot u(z)) = Fu(z)$ is in $\mathcal{P}$ for all $z\in P'$ by checking

$$Fu(z) - u(Fz) \equiv 0 \pmod{P_{c,d,R}} \tag{8.31}$$

in the same way as the proof of (8.28). Thus $\mathcal{V}^{-1}: \mathcal{Q}^{\sigma} \to \mathcal{P}$ is well-defined. Since clearly $\mathcal{V}^{-1}: \mathcal{Q}^{\sigma} \to \mathcal{P}$ is surjective, we obtain (a).

Let us see that $\mathcal{M} = (\mathcal{P}, \mathcal{Q}, \mathcal{F}, \mathcal{V}^{-1}, \langle\ ,\ \rangle)$ satisfies the required properties. The condition (2) is obviously fulfilled. We define the homomorphism $\mathcal{P}\to M_1$ by sending $V^{-s}v_i$ to $\mathcal{V}^{-s}y_i$ and $u(z)$ ($z\in P'$) and $\mathcal{F}^s x_i$ to 0, and the homomorphism $\tilde\kappa: \mathcal{P}\to N$ by sending $V^{-s}v_i$ to $\overline{v}_{i,s}$ and $u(z)$ to $\overline{u}(\overline{z})$ and $\mathcal{F}^s x_i$ to $\overline{\mathcal{F}^s x_i}$. Then $H^1(\mathcal{D}^\bullet)$ is generated by $u(z)$ ($z\in P'$); by (8.28) and (8.31) the homomorphism $H^1(\mathcal{D}^\bullet)\to M'_R$ sending $u(z)$ to z is an isomorphism, i.e., we obtain (1). Next let us show that κ is an isomorphism. This follows from the construction of $\mathcal{M}$; indeed compare (8.16)&(8.20) and (8.13)&(8.22) and (8.12)&(8.24) and (8.15)&(8.25) respectively, and note that these equations determine the isomorphism classes of Z_R and $\mathcal{M}/I_R\mathcal{M}$ respectively. The last property (3) is obviously satisfied.

Finally let us show the last assertion. We assume that $\rho: M(\xi)_R \to \mathcal{M}$ is trivial and show that $\mathcal{C}^\bullet$ is trivial. By the assumption we can write $\rho = \rho_{0,R}$, where $\rho_0 = \rho_k : M(\xi)_k \to M$ with $M := \mathcal{M}_k$ and $\mathcal{M} = M_R$. Write $\phi_0 = \phi_k$ and $f_0 = f_k$, and $\kappa_0 = \kappa_k$. By the property (2), we have $\phi^t = \phi^t_{0,R}$. According to (3), we obtain $f^\vee = \tilde\kappa\circ f^\vee_{0,R}$ with $\tilde\kappa = \kappa\circ\kappa_{0,R}^{-1}\in \operatorname{Aut}(Z_R)$. Then by definition $\mathcal{C}^\bullet$ is trivial (see Prop. 7.6.1, (2)). □

Lemma 8.3.2. *There exist a finite surjective morphism* $\operatorname{Spec}(R)\to\operatorname{Spec}(\Lambda)$ *and elements* $\overline{v}_{i,s}$ *of* N_R *(*$i\in\mathbb{Z}/d\mathbb{Z}$, $0\le s\le\alpha_i$*) with* $\overline{v}_{i,s}\in C_R$ *(*$i\in\mathbb{Z}/d\mathbb{Z}$, $0\le s<\alpha_i$*) such that* $f(\overline{v}_{i,s}) = \overline{\mathcal{V}^{-s}y_i}$ *and* $\dot\varphi^{-1}(\overline{v}_{i,s+1}) = 1\otimes\overline{v}_{i,s}$ *and* $\langle\overline{v}_{i,s}, \overline{v}_{i',s'}\rangle = 0$ *for all* $i,i'\in\mathbb{Z}/d\mathbb{Z}$ *and for all* $0\le s\le\alpha_i$ *and* $0\le s'\le\alpha_{i'}$.

Proof. There exists a finite Λ-algebra Λ' such that $(\Lambda')^{p^{\max_i\{\alpha_i\}}} = \Lambda$. It is possible to choose elements $\overline{v}'_{i,s}$ of $N_{\Lambda'}$ such that $f(\overline{v}'_{i,s}) = \overline{\mathcal{V}^{-s}y_i}$ and $\dot\varphi^{-1}(\overline{v}'_{i,s+1}) = 1\otimes\overline{v}'_{i,s}$. Over an Λ'-algebra R' determined later, we will find $\overline{v}_{i,s}$ of the form

$$\overline{v}_{i,s} = \begin{cases} \overline{v}'_{i,\alpha_i} + \displaystyle\sum_{j\in\mathbb{Z}/d\mathbb{Z}}\sum_{0\le k\le\alpha_j} a_{i,j,k}\cdot\varphi^k\overline{x}_j & \text{for } s=\alpha_i,\\ \overline{v}'_{i,\alpha_i-1} + \displaystyle\sum_{j\in\mathbb{Z}/d\mathbb{Z}} b_{i,j}\cdot\dot\varphi^{-1}\overline{x}_j & \text{for } s=\alpha_i-1,\\ \overline{v}'_{i,s} & \text{for } s<\alpha_i-1,\end{cases}$$

where $a_{i,j,k}$ and $b_{i,j}$ are elements of R' with

$$b^p_{i,j} = a_{i,j,0}. \tag{8.32}$$

These $\overline{v}_{i,s}$ satisfy the two properties $f(\overline{v}_{i,s}) = \overline{\mathcal{V}^{-s}y_i}$ and $\dot{\varphi}^{-1}(\overline{v}_{i,s+1}) = 1 \otimes \overline{v}_{i,s}$.

Using $\dot{\varphi}^{-1}\overline{x}_j = \varphi^{\alpha_{j-1}}\overline{x}_{j-1}$, the condition $\langle \overline{v}_{i,s}, \overline{v}_{j,\alpha_j}\rangle = 0$ is written as

$$\begin{cases} a_{j,i,\alpha_i} - a_{i,j,\alpha_j} = \langle \overline{v}'_{i,\alpha_i}, \overline{v}'_{j,\alpha_j}\rangle & \text{for } s = \alpha_i, \\ a_{j,i,\alpha_i-1} - b_{i,j+1} = \langle \overline{v}'_{i,\alpha_i-1}, \overline{v}'_{j,\alpha_j}\rangle & \text{for } s = \alpha_i - 1, \\ a_{j,i,s} = \langle \overline{v}'_{i,s}, \overline{v}'_{j,\alpha_j}\rangle & \text{for } s < \alpha_i - 1. \end{cases} \tag{8.33}$$

Thus using (8.32) we regard (8.33) as simultaneous equations in $a_{i,j,s}$ $(i, j \in \mathbb{Z}/d\mathbb{Z}$, $0 < s \leq \alpha_j)$ and $b_{i,j}$ $(i, j \in \mathbb{Z}/d\mathbb{Z})$. By Lem. 8.1.2 and Rem. 8.1.3 there exists R' with finite surjective morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(\Lambda')$ such that there exists a solution of the simultaneous equations. Finally we can choose R'-algebra R with finite surjective morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ such that $\overline{v}_{i,s} \in C_R$ $(i \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq s < \alpha_i)$. Then for $0 \leq s < \alpha_i$ and $0 \leq s' < \alpha_{i'}$ we have $\langle \overline{v}_{i,s}, \overline{v}_{i',s'}\rangle = 0$. □

Here is a corollary to Prop. 8.3.1.

Corollary 8.3.3. *Let $\mathcal{C}^\bullet$ be as in (8.8). Assume $\Lambda = k$. Let w be the final element related to Z and let w' be the final element related to $H^1(\mathcal{C}^\bullet)$. Then we have $(d, c) + \xi(w') + (c, d) \prec \xi(w)$.*

Proof. Let M' be a principally quasi-polarized display with Newton polygon $\xi(w')$. Apply Prop. 8.3.1 to this M', we obtain a principally quasi-polarized display $\mathcal{M}$ with Newton polygon $(d, c) + \xi(w') + (c, d)$. By the definition of $\xi(w)$, we have $(d, c) + \xi(w') + (c, d) \prec \xi(w)$. □

8.4. **Proof of Th. 6.1.1.** We use the notation of §6.2. Let

$$\mathcal{C}_0^\bullet : \quad 0 \longrightarrow Z_1^\vee \xrightarrow{f_0^\vee} Z \xrightarrow{f_0} Z_1 \longrightarrow 0$$

be as in (6.2). Put $Z'_0 = H^1(\mathcal{C}_0^\bullet)$, which is a polarized F-zip. Let w'_0 be the final element of Z'_0. Then

Lemma 8.4.1. $(d, c) + \xi(w'_0) + (c, d) = \xi(w)$.

Proof. By Cor. 8.3.3, we have $(d, c) + \xi(w'_0) + (c, d) \prec \xi(w)$. Let X' be the H^1 of the complex (6.1). Clearly $\mathrm{Fz}(X') = Z'_0$. By the definition of $\xi(w'_0)$, we have $\mathcal{N}(X') \prec \xi(w'_0)$. Hence $\xi(w) = \mathcal{N}(X) = (d, c) + \mathcal{N}(X') + (c, d) \prec (d, c) + \xi(w'_0) + (c, d)$. □

We say $\mathcal{C}_0^\bullet$ *splits* if there exists a splitting g_0 of f_0 so that g_0 and $g_0^\vee$ make an isomorphism between Z and $Z_1^\vee \oplus Z'_0 \oplus Z_1$ as polarized F-zips. If $(c, d) = (1, 0)$, then $\mathcal{C}_0^\bullet$ splits. Hence in the non-split case, we have $d > 0$.

We show Th. 6.1.1 by induction on g. The proof is divided into three cases.

Split case: Assume that $\mathcal{C}_0^\bullet$ splits. Let w_1 be the final element of $X_1[p] \times X_1^t[p]$ and let w'_0 be the final element of $Z'_0[p]$. Recall the assumptions: w is not minimal and $\mathcal{C}_0^\bullet$ splits. Then w'_0 is not minimal, since w_1 is minimal (Prop. 6.3.1). Then by the hypothesis of the induction (i.e., Th. 6.1.1 for the lower dimensional case), there exists a non-trivial isogeny

$$H(\xi(w'_0)) \times S \longrightarrow \mathcal{X}'$$

over S of finite type over k with $\dim S > 0$ satisfying the three properties in Th. 6.1.1 for a certain section $\mathrm{Spec}(k) \to S$. Since $\xi(w) = \xi(w_1) + \xi(w'_0)$ (Lem. 8.4.1), the principally quasi-polarized p-divisible group $\mathcal{X} := X_{1,S}^t \oplus \mathcal{X}' \oplus X_{1,S}$ over S satisfies the properties in Th. 6.1.1.

Non-split case (I): Assume that $\mathcal{C}_0^\bullet$ does not split and that Z_0' is not minimal. By the hypothesis of induction (i.e., Th. 6.1.1 for the lower dimensional case), there exists a non-trivial family of isogenies

$$\bigoplus_{l=2}^{\mathfrak{t}-1} H_{m_l,n_l} \otimes R' \longrightarrow \mathcal{X}'$$

over R' such that $\mathrm{Fz}(\mathcal{X}') \simeq Z_{w_0'} \otimes R'$. Then by Prop. 8.3.1 there exists a non-trivial family of self-dual complexes

$$0 \longrightarrow X_1^t \longrightarrow \mathcal{X} \longrightarrow X_1 \longrightarrow 0$$

over R of finite type over k with surjection $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ and a non-trivial family of isogenies

$$\bigoplus_{l=1}^{\mathfrak{t}} H_{m_l,n_l} \otimes R \longrightarrow \mathcal{X}$$

such that $\mathrm{Fz}(\mathcal{X}) \simeq Z_w \otimes R$.

Non-split case (II): Assume that $\mathcal{C}_0^\bullet$ does not split and that Z_0' is minimal. Set $w_0' = \mathcal{E}(Z_0')$. Then w_0' is the minimal final element of Newton polygon $\xi' = \sum_{l=2}^{\mathfrak{t}-1}(m_l, n_l)$. By Prop. 7.6.1 we have a non-trivial family over a ring R' of finite type over k:

$$\mathcal{C}^\bullet: \quad 0 \longrightarrow Z_{1,R'}^\vee \xrightarrow{f^\vee} Z_{R'} \xrightarrow{f} Z_{1,R'} \longrightarrow 0$$

such that $\mathcal{C}^\bullet \otimes k = \mathcal{C}_0^\bullet$. If necessary, we shrink R' so that R' will be irreducible of dimension > 0 and $\{\mathcal{E}(H^1(\mathcal{C}^\bullet)_s) | s \in \mathrm{Spec}\, R'\}$ will consist of at most two final elements, say w_0' at a special point and w' at the generic point.

Case $w' = w_0'$: In this case for a faithfully flat finite extension $R' \to R''$ we have $H^1(\mathcal{C}^\bullet) \otimes R'' \simeq \mathrm{Fz}(H(\xi')) \otimes R''$ (see [21], Cor. 5.4). Set $\mathcal{X}' = H(\xi') \otimes R''$. By Prop. 8.3.1, there exists a non-trivial family over R of finite type over k with some surjection $\mathrm{Spec}(R) \to \mathrm{Spec}(R'')$:

$$\mathcal{D}^\bullet: \quad 0 \longrightarrow X_1^t \longrightarrow \mathcal{X} \longrightarrow X_1 \longrightarrow 0$$

(satisfying $H^1(\mathcal{D}^\bullet) = \mathcal{X}'$) with non-trivial family of isogenies

$$\bigoplus_{l=1}^{\mathfrak{t}} H_{m_l,n_l} \otimes R \longrightarrow \mathcal{X}$$

such that $\mathrm{Fz}(\mathcal{X}) \simeq Z_R$.

Case $w' \neq w_0'$: First we prove

Lemma 8.4.2. *$\xi(w_0') = \xi(w')$.*

Proof. By [27], (4.11) we have $\xi(w_0') \prec \xi(w')$. On the other hand, since $(d,c) + \xi(w') + (c,d) \prec \xi(w)$ by Cor. 8.3.3 and $(d,c) + \xi(w_0') + (c,d) = \xi(w)$ by Lem. 8.4.1, we have $\xi(w') \prec \xi(w_0')$. □

This lemma and $w' \neq w_0'$ imply that w' is not minimal. Take a point $x' \in (\mathcal{W}_{\xi(w')} \cap \mathcal{S}_{w'})(k)$ and let A' be the associated principally polarized abelian variety. Put $Y' = A'[p^\infty]$. Applying Prop. 8.3.1 to $\mathcal{C}^\bullet \otimes_{R'} k'$ for an algebraically closed field k' containing R', there exists a self-dual complex over k'

$$0 \longrightarrow X_{1,k'}^t \longrightarrow Y \longrightarrow X_{1,k'} \longrightarrow 0 \tag{8.34}$$

with $\mathcal{N}(Y) = \xi(w)$ and $\mathcal{E}(\mathrm{Fz}(Y)) = w$ such that the first cohomology of (8.34) is $Y'_{k'}$. Replacing X by Y, we can reduce to the non-split case **(I)**.

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