On p-divisible groups with saturated Newton polygons

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Abstract

This paper concerns the classification of isogeny classes of p-divisible groups with saturated Newton polygons. Let S be a normal noetherian scheme in positive characteristic p with a prime Weil divisor D. Let $\mathcal{X}$ be a p-divisible group over S whose geometric fibers over $S \setminus D$ (resp. over D) have the same Newton polygon. Assume that the Newton polygon of $\mathcal{X}_D$ is saturated in that of $\mathcal{X}_{S\setminus D}$. Our main result (Corollary 1.1) says that $\mathcal{X}$ is isogenous to a p-divisible group over S whose geometric fibers are all minimal. As an application, we give a geometric proof of the unpolarized analogue of Oort's conjecture [11, 6.9].

1. Introduction

Let S be a scheme in positive characteristic p. A p-divisible group over S is called *NP-constant* if all its geometric fibers have the same Newton polygon. In [19] Zink proved that if S is regular, then any NP-constant p-divisible group over S is isogenous to a p-divisible group which has a slope filtration. The case that S is finitely generated over a perfect field with $\dim(S) = 1$ had already been shown by Katz [7, Corollary 2.6.3]. The result of Oort and Zink [15, Theorem 2.1] is quite general, where they showed that the same statement holds even when S is a normal noetherian scheme.

The aim of this paper is to weaken the NP-constancy condition. Since the condition on slope filtration makes sense only for NP-constant p-divisible groups, we instead use the condition that all geometric fibers are minimal. The definition of minimality of [10, 1.1] is recalled in Definition 3.4. Note that

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any NP-constant p-divisible group whose geometric fibers are all minimal has a slope filtration.

Let S be a scheme in characteristic $p > 0$, and let D be a closed subscheme on S. An *NP-saturated p-divisible group over* (S, D) is a p-divisible group $\mathcal{X}$ over S such that $\mathcal{X}_{S \setminus D}$ and $\mathcal{X}_D$ are NP-constant and the Newton polygon of $\mathcal{X}_D$ is saturated in that of $\mathcal{X}_{S \setminus D}$. Here for two Newton polygons ξ, ζ where ζ is less than ξ, we say that ζ is saturated in ξ if there is no other Newton polygon between ζ and ξ. As a corollary of our main theorem (Theorem 4.2), we have

Corollary 1.1. *Assume that S is noetherian and normal and that D is a prime Weil divisor. Then any NP-saturated p-divisible group over (S, D) is isogenous to a p-divisible group over S whose geometric fibers are all minimal.*

This means that in order to classify up to isogeny, NP-saturated p-divisible groups over (S, D) as in Corollary 1.1, it suffices to look into NP-saturated p-divisible groups whose geometric fibers are all minimal. Such p-divisible groups are very specific, which can be said to be concrete objects in the deformation theory at least for local S, since the isomorphism class of every geometric fiber is determined.

This paper is organized as follows. In Section 2 we introduce the notion of quasi-saturated Newton polygons. The above corollary will be regarded as a special case of more general result on NP-quasi-saturated p-divisible groups. In Section 3 we investigate the relation between the slope-divisibility and the minimality of p-divisible groups, and introduce an isogeny $\theta_\mu : X \to \Psi_\mu(X)$ in (18) and show some nice properties of the isogeny, which will be used in the next section. The former part of Section 4 is the heart of this paper, where we shall prove the theorem in the case of $S = \mathrm{Spec}(R)$ with discrete valuation ring R. In the latter part we shall extend it to general (S, D) as in Corollary 1.1, using the ideas invented by [15]. In Section 5, as an application, we give a geometrical proof of the unpolarized analogue of [2, Corollary 3.2] on the configuration of the minimal p-kernel type, and show the unpolarized analogue of Oort's conjecture.

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2. Quasi-saturated Newton polygons

A Newton polygon is a finite multiset of coprime pairs of non-negative integers

$$\{(m_1, n_1), \ldots, (m_t, n_t)\}, \tag{1}$$

i.e., a function from the set of coprime pairs of non-negative integers to the set of non-negative integers with finite support. We define the addition of Newton polygons to be the addition of their functions, which will be denoted by $+_{\mathrm{NP}}$ so that we distinguish this from addition of two-dimensional vectors.

We regard Newton polygons as **upward**-convex line graphs defined in the following way. Let $\xi = \{(m_1, n_1), \ldots, (m_t, n_t)\}$ be a Newton polygon. Put $h = \sum_{i=1}^{t}(m_i + n_i)$ and $d = \sum_{i=1}^{t} n_i$. Set $\lambda_i = n_i/h_i$ with $h_i := m_i + n_i$. We arrange the coprime pairs (m_i, n_i) $(i = 1, \ldots, t)$ so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t.$$

To ξ we associate the line graph obtained as the upper convex hull of the points $\sum_{i=1}^{j}(h_i, n_i)$ for $j = 0, \ldots, t$. The line graph starts at $(0, 0)$ and ends at (h, d). We call (h_i, n_i) $(i = 1, \ldots, t)$ *segments* of ξ.

Let ξ be a Newton polygon. If a point P is **below** or on ξ, we write $P \preceq \xi$. For another Newton polygon ζ whose end point is equal to that of ξ, we say $\zeta \preceq \xi$ if for every point P on ζ we have $P \preceq \xi$. We say $\zeta \prec \xi$ if $\zeta \preceq \xi$ and $\zeta \neq \xi$. Let ζ and ξ be Newton polygons with $\zeta \prec \xi$. We say that $\zeta \prec \xi$ is *saturated* if there is no Newton polygon η such that $\zeta \prec \eta \prec \xi$.

In the rest of this section, we introduce the notion of quasi-saturated pairs of Newton polygons, for which almost all arguments in this paper work, and give a numerical criterion for the saturatedness in the case that ξ consists of two segments, see Lemma 2.2 below.

To a rational number $\lambda = r/s$ with coprime non-negative integers r, s, we associate the two-dimensional vectors

$$v_\lambda = (s, r) \tag{2}$$

and

$$\alpha_\lambda(\xi) = \sum_{n_i/h_i > \lambda} (h_i, n_i) \tag{3}$$

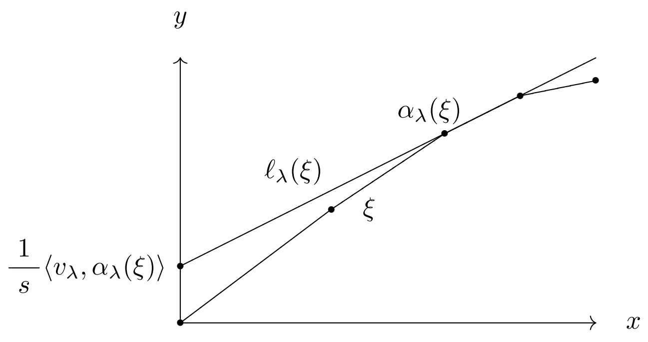
for a Newton polygon ξ of the form (1). We use the alternating form $\langle\ ,\ \rangle$ on two-dimensional vectors:

$$\langle (a, b), (c, d) \rangle = ad - bc. \tag{4}$$

If $\zeta \preceq \xi$, then we have $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta) \rangle \geq 0$ for any λ. This is clear if we know the following graphical meaning of the value $\langle v_\lambda, \alpha_\lambda(\xi) \rangle$: for $v_\lambda = (s, r)$, the line with slope r/s which is tangent to ξ is given by

$$\ell_\lambda(\xi): \quad y = \frac{r}{s}x + \frac{1}{s}\langle v_\lambda, \alpha_\lambda(\xi)\rangle.$$

Note that $\alpha_\lambda(\xi)$ is the first point where $\ell_\lambda(\xi)$ is tangent to ξ.



If $\zeta \prec \zeta$, then $\ell_\lambda(\zeta)$ is below or on $\ell_\lambda(\xi)$, whence $\langle v_\lambda, \alpha_\lambda(\zeta)\rangle \le \langle v_\lambda, \alpha_\lambda(\xi)\rangle$.

Definition 2.1. We say that $\zeta \preceq \xi$ is *quasi-saturated* if for each slope λ of ζ we have $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta)\rangle \le 1$.

Note that the condition of $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta)\rangle \le 1$ is equivalent to that there is no lattice point properly between $\ell_\lambda(\xi)$ and $\ell_\lambda(\zeta)$.

Lemma 2.2. *If $\zeta \prec \xi$ is saturated, then $\zeta \prec \xi$ is quasi-saturated. The converse holds if ξ consists of two segments.*

Proof. Let $\zeta \prec \xi$ be a saturated pair of Newton polygons. One can write

$$\zeta = \varrho +_{\mathrm{NP}} \zeta' \quad \text{and} \quad \xi = \varrho +_{\mathrm{NP}} \xi' \tag{5}$$

so that $\zeta' \prec \xi'$ is saturated and ξ' consists of only two segments. Write

$$\zeta' = \{(m_1, n_1), \ldots, (m_t, n_t)\} \quad \text{and} \quad \xi' = \{(a_1, b_1), (a_2, b_2)\}. \tag{6}$$

Note that ζ' and ϱ do not share any slope. For each slope λ of ϱ we have $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta)\rangle = 0$.

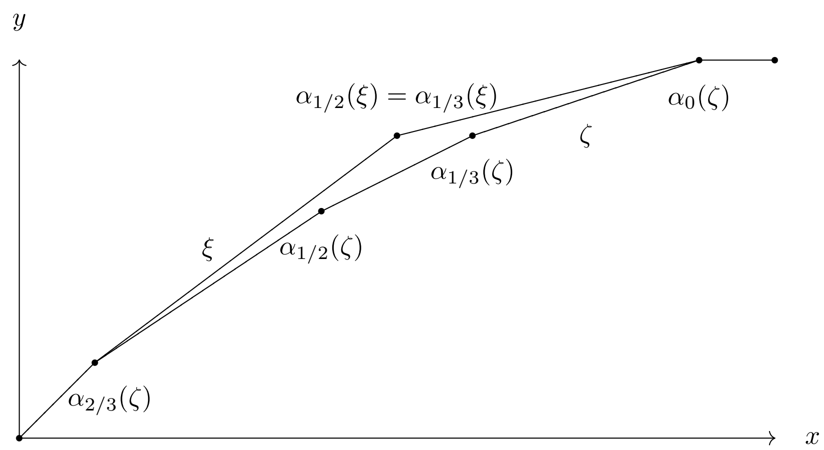
Let λ be a slope of ζ'. Let j be the smallest index with $\lambda = n_j/h_j$ with $h_j = m_j + n_j$. Note $v_\lambda = (h_j, n_j)$. Put $v = (a_1 + b_1, b_1)$ and $u_i = (h_i, n_i)$, which are considered as two-dimensional vectors. We have

$$\alpha_\lambda(\xi) = \alpha_\lambda(\varrho) + v \quad \text{and} \quad \alpha_\lambda(\zeta) = \alpha_\lambda(\varrho) + \sum_{i<j} u_i. \tag{7}$$

The condition $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta)\rangle = 1$ is equivalent to the condition that in the triangle with vertices $v, \sum_{i<j} u_i$ and $\sum_{i\le j} u_i$, there is no lattice point

other than the vertices (in this case the same thing holds for the triangle with vertices $v, \sum_{i<l} u_j$ and $\sum_{i\leq l} u_i$ for all l with $n_l/h_l = \lambda$). Hence the condition that $\langle v_\lambda, \alpha_\lambda(\xi) - \alpha_\lambda(\zeta)\rangle = 1$ for all slopes λ of ζ' is equivalent to that there is no lattice point P above ζ' with $P \preceq \xi'$ except the breaking point of ξ'. This is equivalent to that $\zeta' \prec \xi'$ is saturated. $\square$

Example 2.3. Let $\xi = (0,1) +_{\mathrm{NP}} (1,3) +_{\mathrm{NP}} (3,1) +_{\mathrm{NP}} (1,0)$ and $\zeta = (0,1) +_{\mathrm{NP}} (1,2) +_{\mathrm{NP}} (1,1) +_{\mathrm{NP}} (2,1) +_{\mathrm{NP}} (1,0)$. Note $\zeta \prec \xi$ is saturated. In the proof of the lemma above, we use the notation: $\xi' = (1,3) +_{\mathrm{NP}} (3,1)$ and $\zeta' = (1,2) +_{\mathrm{NP}} (1,1) +_{\mathrm{NP}} (2,1)$ with $\rho = (0,1) +_{\mathrm{NP}} (1,0)$. The picture of $\zeta \prec \xi$ is



In the second statement in Lemma 2.2, the condition that ξ consists of two segments is necessary:

Example 2.4. Consider $\xi = (0,1) +_{\mathrm{NP}} (1,1) +_{\mathrm{NP}} (1,0)$ and $\zeta = 2(1,1)$. Then $\zeta \prec \xi$ is not saturated, since $\zeta \prec (0,1) +_{\mathrm{NP}} (2,1) \prec \xi$. But $\zeta \prec \xi$ is quasi-saturated.

3. Slope-divisibility and minimality

A *slope with exponent* is a pair (λ, e) of rational number λ with $0 \leq \lambda \leq 1$ and integer $e \neq 0$. Let Λ be the set of slopes with exponents:

$$\Lambda = \{(\lambda, e) \in \mathbb{Q} \times \mathbb{Z} \mid 0 \leq \lambda \leq 1,\ e \neq 0\}.$$

For $\mu = (\lambda, e) \in \Lambda$, we call e the *exponent* of μ and λ the *slope* of μ, which will be denoted by $\overline{\mu}$

$$\overline{\mu} := \lambda. \tag{8}$$

Let Λ_e be the subset of Λ consisting of elements with exponent e. We identify Λ_1 with $\{\lambda \in \mathbb{Q} \mid 0 \leq \lambda \leq 1\}$ the set of usual slopes, by mapping $(\lambda, 1)$ to λ.

Let Λ_+ (resp. Λ_-) be the subset of Λ consisting of elements with positive (resp. negative) exponents. We use the embedding of Λ into $\mathbb{Z}^2$ sending $\mu = (r/s, e)$ with coprime integers $r, s \geq 0$ to

$$v_\mu = e(s, r). \tag{9}$$

Let S be a scheme in characteristic $p > 0$. Let $\mathrm{Frob}_S : S \to S$ be the Frobenius morphism. Let X be a p-divisible group over S. Set $X^{(p^a)} = X \times_{\mathrm{Frob}_S^a} S$. We denote by $\mathrm{Fr} : X \to X^{(p)}$ the relative Frobenius homomorphism and by $\mathrm{Ver} : X^{(p)} \to X$ the Verschiebung.

For $\mu = (\lambda, e) \in \Lambda$, we write $\lambda = r/s$ with coprime integers $r, s \geq 0$, and consider the quasi-isogeny

$$\phi_\mu = \left(p^{-r}\mathrm{Fr}^s\right)^e$$

from X to $X^{(p^{se})}$ if $e > 0$ and from $X^{(p^{se})}$ to X if $e < 0$. This will be simply referred as "ϕ_μ on X".

Definition 3.1. Let $\mu \in \Lambda$. We say that X is *slope divisible* (resp. *isoclinic and slope divisible*) with respect to μ if the quasi-isogeny ϕ_μ on X is an isogeny (resp. isomorphism), where $X = 0$ is allowed.

Remark 3.2. If X is slope divisible with respect to $\mu = (\lambda, e)$, then its Serre dual is slope divisible with respect to $(1-\lambda, -e)$. Because the dual of $p^{-r}\mathrm{Fr}^s$ on X is $p^{-r}\mathrm{Ver}^s = \left(p^{-(s-r)}\mathrm{Fr}^s\right)^{-1}$ on the Serre dual. In general when we consider Ver-slopes, negative exponents appear naturally.

For $\mu = (\lambda, e) \in \Lambda$, we set $\mu^* := (\lambda, -e)$. Note $\phi_{\mu^*} = \phi_\mu^{-1}$.

Definition 3.3. Let $\mu \in \Lambda_+$. Let $Y \subset X$ be a closed immersion of p-divisible groups. We say that Y in X is *slope bi-divisible* with respect to μ if the quasi-isogeny ϕ_μ on Y is an isogeny and also ϕ_{μ^*} on X/Y is an isogeny.

Let $\mathbb{D}$ be the covariant Dieudonné functor with $\mathbb{D}(\mathrm{Fr}) = V$ and $\mathbb{D}(\mathrm{Ver}) = F$. Let m and n be coprime non-negative integers. Let $H_{m,n}$ be the p-divisible group over $\mathbb{F}_p$ whose Dieudonné module $N_{m,n} = \mathbb{D}(H_{m,n})$ is given by

$$N_{m,n} = \bigoplus_{i=1}^{m+n} \mathbb{Z}_p \epsilon_i \tag{10}$$

with $F\epsilon_i = \epsilon_{i+m}$ and $V\epsilon_i = \epsilon_{i+n}$ and $\epsilon_{i+m+n} = p\epsilon_i$. Note that $H_{m,n}$ is a simple p-divisible group with slope $n/(m+n)$. Let ϖ be the endomorphism of $H_{m,n}$ characterized by $\mathbb{D}(\varpi)(\epsilon_i) = \epsilon_{i+1}$. It is straightforward to see

$$\phi_\mu = \varpi^{\langle v_\mu, (m+n,n)\rangle}. \tag{11}$$

Let K be a field of characteristic p. A p-divisible group over K is called *isoclinic and minimal* if it is isomorphic over the algebraic closure $\overline{K}$ of K to the product of some copies of $(H_{m,n})_{\overline{K}}$ for a certain coprime pair (m,n) of non-negative integers. Clearly an isoclinic minimal p-divisible group with slope λ is slope divisible with respect to any μ with $\langle v_\mu, v_\lambda \rangle \geq 0$ and is isoclinic and slope divisible with respect to λ.

Recall the definition [10, 1.1] of minimal p-divisible groups. For a Newton polygon $\xi = \{(m_1,n_1),\dots,(m_t,n_t)\}$, we set

$$H(\xi) := \bigoplus_{i=1}^{t} H_{m_i,n_i}. \tag{12}$$

Definition 3.4. A p-divisible group over K is called *minimal* if it is isomorphic over $\overline{K}$ to $H(\xi)_{\overline{K}}$ for some Newton polygon ξ.

Also recall the definition of completely slope divisible p-divisible groups, which is slightly generalized from that in [15, 1.2] for later use. Let us introduce a "partial" variant at the same time.

Definition 3.5. Let $\mu_1,\dots,\mu_\ell \in \Lambda_+$ with $\overline{\mu_1} > \cdots > \overline{\mu_\ell}$. A p-divisible group X is called *partially completely slope divisible with respect to* $\mu_1,\dots,\mu_\ell$ if there exists a filtration by closed immersions of p-divisible groups

$$0 \subset X_0 \subset X_1 \subset \cdots \subset X_{\ell-1} \subset X_\ell = X \tag{13}$$

such that

(i) X_j $(j \leq i)$ are slope divisible with respect to μ_i for $1 \leq i \leq \ell$;

(ii) $\mathrm{Gr}_i(X) := X_i/X_{i-1}$ is isoclinic and slope divisible with respect to μ_i for $1 \leq i \leq \ell$;

(iii) all the slopes of X_0 are greater than $\overline{\mu_1}$.

When $X_0 = 0$, we remove "partially".

By the same way as in [19, Corollary 11], one can show

Lemma 3.6. *Assume that K is a perfect field of characteristic p. Let X be a partially completely slope divisible p-divisible group over K with respect to $\mu_1,\dots,\mu_\ell$. Then X is isomorphic to $X_0 \oplus \bigoplus_{i=1}^{\ell} \mathrm{Gr}_i(X)$.*

Let us define the bi-divisible variant.

Definition 3.7. Let $\mu_1,\dots,\mu_\ell \in \Lambda_+$ with $\overline{\mu_1} > \cdots > \overline{\mu_\ell}$. A p-divisible group X is called *partially completely slope bi-divisible with respect to* $\mu_1,\dots,\mu_\ell$ if there exists a filtration by closed immersions of p-divisible groups

$$0 \subset X_0 \subset X_1 \subset \cdots \subset X_{\ell-1} \subset X_\ell = X \tag{14}$$

such that

(i) X_j ($j \le i$) are slope divisible with respect to μ_i for $1 \le i \le \ell$;

(ii) X/X_j ($j \ge i-1$) are slope divisible with respect to μ_i^* for $1 \le i \le \ell$;

(iii) all the slopes of X_0 are greater than $\overline{\mu_1}$.

When $X_0 = 0$, we remove "partially".

Lemma 3.8. *Let X be a p-divisible group. Assume that X is partially completely slope bi-divisible with respect to $\mu_1, \dots, \mu_\ell$. Then*

(1) *X is partially completely slope divisible with respect to $\mu_1, \dots, \mu_\ell$.*

(2) *For $i = 1, \dots, \ell$, we have that $\mathrm{Gr}_i(X)$ is slope divisible with respect to μ_a for $a \ge i$ and μ_b^* for $b \le i$.*

Proof. (1) The quasi-isogeny $\phi_{\mu_i} : X_i/X_{i-1} \to X_i/X_{i-1}$ is an isogeny, because this is induced by the isogeny ϕ_{μ_i} on X_i. Consider the composition $X_i/X_{i-1} \to X_i/X_{i-1} \to X/X_{i-1}$ of ϕ_{μ_i} and the restriction to X_i/X_{i-1} of $\phi_{\mu_i^*}$ on X/X_{i-1}. Since $\phi_{\mu_i^*} = \phi_{\mu_i}^{-1}$, this composition is identical on X_i/X_{i-1}. In particular the kernel of $\phi_{\mu_i} : X_i/X_{i-1} \to X_i/X_{i-1}$ is zero. Hence X_i/X_{i-1} is isoclinic and slope divisible with respect to μ_i.

(2) It suffices to show this for each geometric fiber. Hence we may assume that X is a p-divisible group over an algebraically closed field. By Lemma 3.6 X is isomorphic to $X_0 \oplus \bigoplus_{i=1}^{\ell} \mathrm{Gr}_i(X)$. Since X_i is slope divisible with respect to μ_a for $a \ge i$, its direct summand $\mathrm{Gr}_i(X)$ is also slope divisible with respect to μ_a for $a \ge i$. Since X/X_{i-1} is slope divisible with respect to μ_b^* for $b \le i$, its direct summand $\mathrm{Gr}_i(X)$ is also slope divisible with respect to μ_b^* for $b \le i$. $\square$

Remark 3.9. Let X be a minimal p-divisible group over a field K of characteristic p. Then X is completely slope bi-divisible with respect to its slopes.

Example 3.10. Let $N_{3,2} = \bigoplus_{i=1}^{5} \mathbb{Z}_p \epsilon_i$ be as in (10). Let M be the Dieudonné submodule of $N_{3,2}$ generated by $\epsilon_1, p\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5$. Let Y be a p-divisible group over $\mathbb{F}_p$ whose Dieudonné module is isomorphic to M. Let $X = H_{1,1} \oplus Y$. Set $\mu_1 = (1/2, 1)$ and $\mu_2 = (2/5, 1)$. Note that X is completely slope divisible with respect to μ_1, μ_2, whose slope filtration is $0 \subset H_{1,1} \subset X$. But X is not completely slope bi-divisible with respect to μ_1, μ_2, since $\phi_{\mu_1^*} = p^{-1}\mathrm{Ver}^2$ is not isogeny on Y.

Lemma 3.11. *Let $\mathcal{X}$ be an NP-constant p-divisible group over S. Then the subset of points of S over which the fiber of $\mathcal{X}$ is completely slope bi-divisible with respect to $\mu_1, \dots, \mu_\ell$ is closed in S.*

Proof. Write $v_{\mu_i} = (s_i, r_i)$. Let s be the least common multiple of $s_1, \dots, s_\ell$. Let μ_i' be the elements of Λ_+ such that $v_{\mu_i'} = (s/s_i)v_{\mu_i}$. By [15, 2.3], the subset of points of S over which the fiber of $\mathcal{X}$ is completely slope divisible with respect to $\mu'_1, \dots, \mu'_\ell$ is closed in S. Then the lemma follows from the fact [17, Proposition 2.9] that for a quasi-isogeny $\rho : X \to Y$ of p-divisible groups over S, the subset of points of S over which ρ is an isogeny is closed in S. $\square$

We have seen in Remark 3.9 that any minimal p-divisible group is completely slope divisible. Let us study when a completely slope divisible p-divisible group is minimal.

Proposition 3.12. *Let $\lambda \in \Lambda_1$. Let X be a p-divisible group over a field K of characteristic p which is isoclinic and slope divisible with respect to λ. The following are equivalent.*

(1) *X is minimal;*

(2) *for any $\mu \in \Lambda$ with $\langle v_\mu, v_\lambda \rangle > 0$, the quasi-isogeny ϕ_μ on X is an isogeny;*

(3) *for a $\mu \in \Lambda$ with $\langle v_\mu, v_\lambda \rangle = 1$, the quasi-isogeny ϕ_μ on X is an isogeny.*

Proof. It suffices to show the case that K is algebraically closed. For $v_\lambda = (m+n, n)$ we write

$$H_\lambda := (H_{m,n})_K. \tag{15}$$

(1)$\Rightarrow$(2): Let X be an isoclinic and minimal p-divisible group, say

$$X = H_\lambda^{\oplus \nu}.$$

Let $\mu \in \Lambda$ with $\langle v_\mu, v_\lambda \rangle > 0$. As seen in (11), $\mathbb{D}(\phi_\mu)$ on $\mathbb{D}(H_\lambda)$ is the map sending ϵ_i to $\epsilon_{i+\langle v_\mu, v_\lambda \rangle}$. Thus ϕ_μ on $H_\lambda^{\oplus \nu}$ is an isogeny.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1): Write $v_\lambda = (m+n, n)$ and $v_\mu = (a+b, b)$. Since $\phi_\lambda = p^{-n}\mathrm{Fr}^{m+n}$ and $\phi_\mu = p^{-b}\mathrm{Fr}^{a+b}$, we have

$$\phi_\lambda^{-b}\phi_\mu^{n} = \mathrm{Fr}, \quad \phi_\lambda^{-a}\phi_\mu^{m} = \mathrm{Ver}. \tag{16}$$

Let $G_i = X[\phi_\mu^i]$ be the kernel of the isogeny ϕ_μ^i on X for $i = 0, 1, \dots, m+n$. We have a filtration of $X[p]$:

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{m+n} = X[p].$$

By (16) we have $\mathrm{Fr}\, G_i = G_{i-n}$ and $\mathrm{Ver}\, G_i = G_{i-m}$. Since m and n are coprime, $\{G_i/G_{i-1} \mid i = 1, \cdots, m+n\}$ consists of one $(\mathrm{Ver}, \mathrm{Fr}^{-1})$-cycle (cf. [8]), whence G_i/G_{i-1} $(i = 1, \cdots, m+n)$ have the same rank, say ν. Thus $X[p]$ is isomorphic to $(H_{m,n}^{\oplus \nu}[p])_K$, and therefore X is minimal by [10]. $\square$

Let us give an alternative proof of a special case of [11], 2.2:

Corollary 3.13. *Let $\mathcal{X}$ be an NP-constant p-divisible group over S. Then the subset of points of S over which the fiber of $\mathcal{X}$ is minimal is closed in S.*

Proof. Let $\lambda_1 > \cdots > \lambda_\ell$ be the slopes of $\mathcal{X}$. By the similar way to that in Lemma 3.11, the subset of points of S over which the fiber of $\mathcal{X}$ is completely slope divisible with respect to $\lambda_1, \dots, \lambda_\ell$ is closed in S. Hence we may assume that $\mathcal{X}$ is completely slope divisible with respect to $\lambda_1, \dots, \lambda_\ell$. Let $0 = \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_\ell = \mathcal{X}$ be the slope filtration. Let s be a point of S. Note that $\mathcal{X}_s$ is minimal if and only if $(\mathcal{X}_i/\mathcal{X}_{i-1})_s$ is minimal for all $i = 1, \dots, \ell$ (cf. Lemma 3.6). By Proposition 3.12, $(\mathcal{X}_i/\mathcal{X}_{i-1})_s$ is minimal if and only if ϕ_μ on $(\mathcal{X}_i/\mathcal{X}_{i-1})_s$ is an isogeny for some $\mu \in \Lambda$ with $\langle v_\mu, v_\lambda \rangle = 1$. Hence the corollary follows from [17, Proposition 2.9]. □

Let K be a field of characteristic p. Recall the definition of the small image of a homomorphism of p-divisible groups over K. This notion was introduced by Zink in [19], §3. Let $g : G \to H$ be a homomorphism of p-divisible groups over K. It is showed in [19], Prop. 8 that g has a unique factorization in the category of p-divisible groups

$$G \to G' \to H' \to H,$$

where $G' \to H'$ is an isogeny, $H' \to H$ is a monomorphism of p-divisible groups and $G \to G'$ is a homomorphism satisfying that $G[p^n] \to G'[p^n]$ is an epimorphism for each natural number n. We call G' the *small image* of g. In the proof of [19], Prop. 8, the small image G' is given by the quotient of G by A', where A' is the unique p-divisible subgroup of $\mathrm{Ker}(g)$ such that $\mathrm{Ker}(g)/A'$ is a finite group scheme. If K is perfect, then $\mathbb{D}(G')$ is the image of $\mathbb{D}(g)$, and $\mathbb{D}(H')$ is the smallest direct summand of $\mathbb{D}(H)$ containing $\mathbb{D}(G')$.

Let X be a p-divisible group over K. Let $\mu \in \Lambda_+$ and write $v_\mu = (s, r)$. Let $\Psi_\mu(X)$ be the small image of

$$f_\mu : \quad X \times X^{(p^s)} \xrightarrow{p^{s-r} \times \mathrm{Ver}^s} X \times X \longrightarrow X, \tag{17}$$

where the second morphism is the addition of X. Let $A = \mathrm{Ker}(f_\mu)$. Consider the homomorphism $g : X^{(p^s)} \to A$ sending y to $(\mathrm{Ver}^s y, -p^{s-r}y)$. The kernel and the cokernel of g are finite, since the both are killed by p^{s-r}. Hence the image Z of g is the maximal p-divisible subgroup of A. By the construction of the small image explained above, we have $\Psi_\mu(X) = X \times X^{(p^s)}/Z$. Composing $(\mathrm{id}, 0) : X \to X \times X^{(p^s)}$ and $X \times X^{(p^s)} \to \Psi_\mu(X)$, we have an isogeny

$$\theta_\mu : \quad X \longrightarrow \Psi_\mu(X). \tag{18}$$

Since the kernel of θ_μ is the intersection of Z and $X \times \{0\}$, we have

Lemma 3.14. *We have*

$$\mathrm{Ker}(\theta_\mu) = \mathrm{Im}(\mathrm{Ver}^s : X^{(p^s)}[p^{s-r}] \to X[p^{s-r}]).$$

Here the right hand side is the image as the fppf sheaf, which is represented by a group scheme $X^{(p^s)}[p^{s-r}]/\operatorname{Ker}\left(\mathrm{Ver}^s_{X^{(p^s)}[p^{s-r}]}\right)$.

Remark 3.15. Assume that K is a perfect field. Let M be the Dieudonné module of X. Then the Dieudonné module of $\Psi_\mu(X)$ is

$$\mathbb{D}(\Psi_\mu(X)) = p^{s-r}M + F^s M,$$

which is isomorphic to $p^{-r}V^s M + M$. The isogeny θ_μ in (18) corresponds to the isogeny $M \to p^{s-r}M + F^s M$ sending m to $p^{s-r}m$. The Dieudonné module of $\mathrm{Ker}(\theta_\mu)$ is

$$(p^{s-r}M + F^s M)/p^{s-r}M,$$

which is isomorphic to $(M + p^{-(s-r)}F^s M)/M$.

Lemma 3.16. *Let* $\mu = (\lambda, e) \in \Lambda_+$. *The following are equivalent.*

(1) $\log_p \deg(\theta_\mu) = 0$.

(2) *X is slope divisible with respect to* $\mu^* = (\lambda, -e)$.

In case, in particular the slopes of X are less than or equal to λ.

Proof. From the above remark, it is clear that (1) is equivalent to $p^{-(s-r)}\mathrm{Ver}^s$ is an isogeny on X. Since $p^{-(s-r)}\mathrm{Ver}^s = (p^{-r}\mathrm{Fr}^s)^{-1}$, we have the lemma. □

For $\mu = (\overline{\mu}, e) \in \Lambda_+$, we set $\alpha_\mu(\xi)$ to be $\alpha_{\overline{\mu}}(\xi)$.

Proposition 3.17. *Let ξ be the Newton polygon of X. Assume that there is a short exact sequence*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

of p-divisible groups over K, which splits over $\overline{K}$. Let $\mu \in \Lambda_+$ such that the slopes of Y are greater than or equal to $\overline{\mu}$ and the slopes of Z are less than or equal to $\overline{\mu}$. Then we have

$$\log_p \deg(\theta_\mu) \geq \langle v_\mu, \alpha_\mu(\xi) \rangle,$$

where the equality holds if and only if Y in X is slope bi-divisible with respect to μ.

Proof. We may assume that K is an algebraically closed field and $X = Y \times Z$. Let h (resp. d) be the height (resp. the dimension) of Y. Then $\langle v_\mu, \alpha_\mu(\xi)\rangle = \langle v_\mu, (h,d)\rangle$. Let M be the Dieudonné module of Y. Set $A = F^s M$ and $B = p^{s-r}M$. Then

$$\log_p \deg(\theta_\mu) = \mathrm{length}(A+B)/B + \log_p \deg(\theta_\mu \text{ on } Z).$$

Lemma 3.16 says that $\log_p \deg(\theta_\mu \text{ on } Z) = 0$ if and only if Z is slope divisible with respect to μ^*. Since $\mathrm{Coker}((A+B)/A \to M/A) = \mathrm{Coker}((A+B)/B \to M/B)$, we have

$$\begin{aligned}
\mathrm{length}(A+B)/B &= \mathrm{length}\, M/B - \mathrm{length}\, M/A + \mathrm{length}(A+B)/A \\
&= (s-r)h - s(h-d) + \mathrm{length}(A+B)/A \\
&= \langle v_\mu, \alpha_\mu(\xi)\rangle + \mathrm{length}(A+B)/A.
\end{aligned}$$

Obviously $(A+B)/A \simeq (M + p^{-r}V^s M)/M$ is zero if and only if Y is slope divisible with respect to μ. □

Lemma 3.18. *Let $\mu \in \Lambda_+$ and $\mu' \in \Lambda$. If X is slope divisible with respect to μ', then $\Psi_\mu(X)$ is slope divisible with respect to μ'. If X is a minimal p-divisible group, then so is $\Psi_\mu(X)$.*

Proof. It suffices to show this over an algebraically closed field. Let M be the Dieudonné module of X. Write $v_\mu = (s,r)$ and $v'_\mu = (s', r')$. Obviously if $p^{-r'}V^{s'}M \subset M$, then $p^{-r'}V^{s'}N \subset N$ for $N = p^{s-r}M + F^s M$. The second assertion follows from Proposition 3.12. □

We collect some basic properties of the operators $\Psi_\mu(-)$ for $\mu \in \Lambda_+$.

Lemma 3.19. *Let X be a p-divisible group over K.*

(1) *We have $\Psi_\mu(\Psi_{\mu'}(X)) = \Psi_{\mu'}(\Psi_\mu(X))$ for $\mu, \mu' \in \Lambda_+$.*

(2) *Let $\mu \in \Lambda_+$. Let*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

be an exact sequence of p-divisible groups over K which splits over $\overline{K}$. Then $Y \to X$ induces a monomorphism $\Psi_\mu(Y) \to \Psi_\mu(X)$ and we have a canonical isomorphism

$$\Psi_\mu(Z) \simeq \Psi_\mu(X)/\Psi_\mu(Y).$$

Proof. (1) Consider the natural isogenies $X \to \Psi_\mu(\Psi_{\mu'}(X))$ and $X \to \Psi_{\mu'}(\Psi_\mu(X))$ (the former one is the composition of $\theta_{\mu'}$ on X and θ_μ on $\Psi_{\mu'}(X)$ and the latter one is obtained by exchanging the roles of μ and μ'). We claim that those kernels are the same. It suffices to see this over $\overline{K}$. Let $M = \mathbb{D}(X_{\overline{K}})$ and set $U_\mu = p^{-r}V^s$ for $v_\mu = (s, r)$. The claim over $\overline{K}$ follows from the equality

$$\mathbb{D}(\Psi_\mu(\Psi_{\mu'}(X_{\overline{K}}))) = M + U_{\mu'}M + U_\mu M + U_\mu U_{\mu'} M = \mathbb{D}(\Psi_{\mu'}(\Psi_\mu(X_{\overline{K}}))).$$

(2) Since the kernel of $\theta_\mu : Y \to \Psi_\mu(Y)$ is contained in the kernel of $\theta_\mu : X \to \Psi_\mu(X)$, we have a homomorphism $\Psi_\mu(Y) \to \Psi_\mu(X)$. It suffices to show that this is a monomorphism over $\overline{K}$. We may assume $X_{\overline{K}} = Y_{\overline{K}} \times Z_{\overline{K}}$. Then

$$\Psi_\mu(X_{\overline{K}}) = \Psi_\mu(Y_{\overline{K}}) \times \Psi_\mu(Z_{\overline{K}}). \tag{19}$$

Hence obviously $\Psi_\mu(Y_{\overline{K}}) \to \Psi_\mu(X_{\overline{K}})$ is a monomorphism.

Note that θ_μ on X and that on Y induce an isogeny $\vartheta : Z \to \Psi_\mu(X)/\Psi_\mu(Y)$. It is enough to show that the kernel of ϑ is the same as the kernel of $\theta_\mu : Z \to \Psi_\mu(Z)$. This follows from the fact that over $\overline{K}$ there is a canonical isomorphism $\Psi_\mu(Z_{\overline{K}}) \simeq \Psi_\mu(X_{\overline{K}})/\Psi_\mu(Y_{\overline{K}})$, which is obtained from (19). □

From now on, for $\mu, \mu' \in \Lambda_+$ we write $\Psi_\mu\Psi_{\mu'}(X)$ for $\Psi_\mu(\Psi_{\mu'}(X))$ and $\Psi_\mu^2(X)$ for $\Psi_\mu(\Psi_\mu(X))$ and so on.

In [19], Lemma 9 and the argument following it, Zink explicitly constructed an isogeny from a given p-divisible group X over K to a p-divisible group which is slope divisible with respect to the smallest slope of X. In the next lemma, we generalize this a little bit for later use.

Lemma 3.20. *Let X be a p-divisible group over K of height h. Let μ be an element of Λ_+ whose slope is less than or equal to the smallest slope of X. Then $\Psi_\mu^{h-1}(X)$ is slope divisible with respect to μ. In particular if X is isoclinic of slope λ, then $\Psi_\mu^{h-1}\Psi_\lambda^{h-1}(X)$ is minimal for $\mu \in \Lambda_+$ with $\langle v_\mu, v_\lambda \rangle = 1$.*

Proof. If suffices to show this over an algebraically closed field. Let M be the Dieudonné module of X. Write $v_\mu = (s, r)$ and set $U_\mu = p^{-r}V^s$. The Dieudonné module $\mathbb{D}(\Psi_\mu^{h-1}(X))$ of $\Psi_\mu^{h-1}(X)$ is isomorphic to

$$M + U_\mu M + \cdots + U_\mu^{h-1} M.$$

We have $U_\mu \mathbb{D}(\Psi_\mu^{h-1}(X)) \subset \mathbb{D}(\Psi_\mu^{h-1}(X))$, since the proof of [19, Lemma 9] works without change. Thus we obtain the first assertion. The second one follows from Lemma 3.18 and Proposition 3.12. □

The following bi-divisible variant of Lemma 3.20 plays an important role in the proof of our main results.

Lemma 3.21. *Let X, Y, Z and μ be as in Proposition 3.17. Let h be the height of X. Then $\Psi_\mu^{h-1}(Y)$ in $\Psi_\mu^{h-1}(X)$ is slope bi-divisible with respect to μ.*

Proof. It is enough to show this over an algebraically closed field. We may assume $X = Y \times Z$. Applying Lemma 3.20 to Y, we have that $\Psi_\mu^{h-1}(Y)$ is slope divisible with respect to μ. It remains to show that $\Psi_\mu^{h-1}(Z)$ is slope divisible with respect to μ^*. Let N be the Dieudonné module of Z. The Dieudonné module of $\Psi_\mu(Z)$ is $p^{-r}V^s N + N$, which is isomorphic to $N + U_{\mu^*}N$ where $U_{\mu^*} = (p^{-r}V^s)^{-1} = p^{-(s-r)}F^s$. Hence the Dieudonné module $\Psi_\mu^{h-1}(Z)$ is isomorphic to

$$M + U_{\mu^*}M + \cdots + U_{\mu^*}^{h-1}M.$$

One can show that this is slope divisible with respect to μ^*, in the same way as in Lemma 3.20, considering Ver-slope in stead of slope (=Fr-slope). $\square$

The next proposition will be used in induction steps when we construct an isogeny from a given p-divisible group over K to a completely slope bi-divisible p-divisible group.

Proposition 3.22. *Let X be a partially completely slope bi-divisible p-divisible group over K with respect to $\mu_2, \dots, \mu_\ell$ with filtration*

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_{\ell-1} \subset X_\ell = X.$$

Let μ_1 be an element of Λ_+ whose slope is greater than $\overline{\mu_2}$ and is less than or equal to the smallest slope of X_1. Let e be a non-negative integer such that $\Psi_{\mu_1}^e X_1$ in $\Psi_{\mu_1}^e X$ is slope bi-divisible with respect to μ_1 (Lemma 3.21 says that $e = h - 1$ satisfies this condition). Then $\Psi_{\mu_1}^e X$ is partially completely slope bi-divisible with respect to $\mu_1, \mu_2, \dots, \mu_\ell$.

Proof. Set $Y := \Psi_{\mu_1}^e X$ and $Y_i := \Psi_{\mu_1}^e X_i$ for $i = 1, \dots, \ell$. As obtained in [19], (11) on p. 89, there is an exact sequence of p-divisible groups

$$0 \longrightarrow Y_1^{\phi_{\mu_1}-\text{nul}} \longrightarrow Y_1 \longrightarrow Y_1^{\phi_{\mu_1}-\text{ét}} \longrightarrow 0, \tag{20}$$

where $Y_1^{\phi_{\mu_1}-\text{ét}}$ and $Y_1^{\phi_{\mu_1}-\text{nul}}$ are characterized by the property that ϕ_{μ_1} induces an isomorphism on $Y_1^{\phi_{\mu_1}-\text{ét}}[p^n]$ and is nilpotent on $Y_1^{\phi_{\mu_1}-\text{nul}}[p^n]$ for all n. Put $Y_0 := Y_1^{\phi_{\mu_1}-\text{nul}}$. We claim that Y is partially completely slope bi-divisible with respect to $\mu_1, \mu_2, \dots, \mu_\ell$ with filtration

$$0 \subset Y_0 \subset Y_1 \subset \cdots \subset Y_\ell = Y.$$

We need to check that this filtration $Y_\bullet$ satisfies the conditions (i), (ii), (iii) in Definition 3.7.

By the definition of Y_0, all the slopes of Y_0 are greater than $\overline{\mu_1}$, whence $Y_\bullet$ satisfies (iii).

As X_j $(j \le i)$ are slope divisible with respect to μ_i for $2 \le i \le \ell$, so are Y_j $(j \le i)$ by Lemma 3.18. By the assumption, Y_1 is slope divisible with respect to μ_1. As X_1 is slope divisible with respect to μ_i ($i \geq 2$), so is $Y_1 = \Psi^e_{\mu_1} X_1$ by Lemma 3.18. Since $(Y_0)_{\overline{K}}$ is a direct summand of $(Y_1)_{\overline{K}}$, we have that Y_0 is slope divisible with respect to μ_i ($i \geq 1$). Hence $Y_\bullet$ satisfies (i).

By Lemma 3.19, (2), the p-divisible group Y/Y_j is isomorphic to $\Psi^e_{\mu_1}(X/X_j)$ for $j \geq 1$. By Lemma 3.18, $Y/Y_j \simeq \Psi^e_{\mu_1}(X/X_j)$ ($j \geq 1$) is slope divisible with respect to μ_i^* for $i \leq j+1$. Over the algebraic closure $\overline{K}$ of K, we have $(Y/Y_0)_{\overline{K}} \simeq (Y_1/Y_0)_{\overline{K}} \oplus (Y/Y_1)_{\overline{K}}$. Since $(Y_1/Y_0)_{\overline{K}}$ and $(Y/Y_1)_{\overline{K}}$ are both slope divisible with respect to μ_1^*, we have that $(Y/Y_0)_{\overline{K}}$ is slope divisible with respect to μ_1^* and therefore so is Y/Y_0. Thus $Y_\bullet$ satisfies (ii). □

Now we get the main result over a field of characteristic p:

Corollary 3.23. *Let X be a p-divisible group over K of height h. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$ be the set of positive slopes of X. Then*

(1) *$(\prod_{i=1}^{\ell} \Psi^{h-1}_{\lambda_i})(X)$ is completely slope bi-divisible with respect to $\lambda_1, \ldots, \lambda_\ell, 0$.*

(2) *We choose $\mu_i \in \Lambda_+$ such that $\langle v_{\mu_i}, v_{\lambda_i} \rangle = 1$. Then $(\prod_{i=1}^{\ell} \Psi^{h-1}_{\mu_i} \Psi^{h-1}_{\lambda_i})(X)$ is minimal.*

Here recall that λ_i is regarded as the element $(\lambda_i, 1)$ of Λ_1 for each $i = 1, \ldots, \ell$.

Proof. It suffices to show these over the algebraic closure of K. Therefore we assume that K is algebraically closed. Then, as it suffices to show them for the formal part of X, we may assume that X is a formal p-divisible group (i.e., every slope of X is positive).

(1) By Proposition 3.22, inductively one can check that $(\prod_{i=j}^{\ell} \Psi^{h-1}_{\lambda_i})(X)$ is partially completely slope bi-divisible with respect to $\lambda_j, \ldots, \lambda_\ell$.

(2) Set $Y := (\prod_{i=1}^{\ell} \Psi^{h-1}_{\lambda_i})(X)$. By (1), Y is completely slope bi-divisible with respect to $\lambda_1, \ldots, \lambda_\ell$. Set $Z := (\prod_{i=1}^{\ell} \Psi^{h-1}_{\mu_i})(Y)$, which is also completely slope bi-divisible with respect to $\lambda_1, \ldots, \lambda_\ell$. Since $\mathrm{Gr}_j(Y)$ is isoclinic and slope divisible with respect to λ_j, so is $\mathrm{Gr}_j(Z) = (\prod_{i=1}^{\ell} \Psi^{h-1}_{\mu_i}) \mathrm{Gr}_j(Y)$ by Lemma 3.18. Also $\mathrm{Gr}_j(Z) = \Psi^{h-1}_{\mu_j}(\prod_{i \neq j} \Psi^{h-1}_{\mu_i} \mathrm{Gr}_j(Y))$ is slope divisible with respect to μ_j by Lemma 3.20. Hence $\mathrm{Gr}_j(Z)$ is minimal by Proposition 3.12 and therefore so is Z. □

4. Proof

We start with proving our main result (Proposition 4.1) over a discrete valuation ring. Based on this result, we shall show the main theorem (Theorem 4.2).

The result over a discrete valuation ring is stated in terms of Raynaud's flat extension. Let R be a discrete valuation ring of characteristic p. Let K be the quotient ring of R. Let $\mathcal{X}$ be a p-divisible group over R. Write $X = \mathcal{X}_K$.

Let G be a finite subgroup scheme of X. This defines an isogeny $\rho : X \to Y$ of p-divisible groups with $G = \ker(\rho)$. Let N be a sufficiently large integer such that $G \subset X[p^N]$. Let $\mathcal{G}$ be the schematic closure in $\mathcal{X}[p^N]$ of G. Note that $\mathcal{G}$ is a flat subgroup scheme of $\mathcal{X}[p^N]$, see [18], p. 259–260. By taking the quotient by $\mathcal{G}$, we have an isogeny $\tilde{\rho} : \mathcal{X} \to \mathcal{Y}$. This construction of the isogeny $\tilde{\rho} : \mathcal{X} \to \mathcal{Y}$ from a given data $(\mathcal{X},\ \rho : X \to Y)$ is called the *flat extension.*

An *NP-quasi-saturated p-divisible group over* (S, D) is defined by replacing "saturated" by "quasi-saturated" in the definition of NP-saturated p-divisible group over (S, D). An NP-quasi-saturated p-divisible group over R is that over (S, D) with $S = \mathrm{Spec}(R)$ and $D = \mathrm{Spec}(k)$, where k is the residue field of R.

Proposition 4.1. *Let $\mathcal{X}$ be an NP-quasi-saturated p-divisible group over R. Set $X = \mathcal{X}_K$. Let ξ (resp. ζ) be the Newton polygon of X (resp. $\mathcal{X}_k$). Let $\{\mu_1, \dots, \mu_\ell\}$ be a subset of Λ_+ containing all slopes of ζ such that $\langle v_{\mu_i}, \alpha_{\mu_i}(\xi) - \alpha_{\mu_i}(\zeta)\rangle \le 1$. Suppose $\overline{\mu_1} > \cdots > \overline{\mu_\ell}$. Then there exists an isogeny $\rho : X \to Y$ over K whose flat extension $\mathcal{X} \to \mathcal{Y}$ satisfies that $\mathcal{Y}_K$ is minimal and $\mathcal{Y}_k$ is completely slope bi-divisible with respect to $\mu_1, \dots, \mu_\ell$. Moreover, the isogeny $\rho : X \to Y$ can be taken as a composition of θ_{μ_i}'s for $1 \le i \le \ell$, see (18) for the definition of θ_μ.*

Proof. We first reduce to the case where X is minimal. If the theorem is true for minimal X', choose an isogeny $X \to X'$ with X' minimal (Corollary 3.23, (2)), let $X' \to Y$ be an isogeny obtained from the theorem for X'; then the composition $\rho : X \to X' \to Y$ satisfies the properties of the theorem.

So we assume that X is minimal. It suffices to show that if $\mathcal{X}_{\overline{k}}$ is partially completely slope bi-divisible with respect to $\mu_{i+1}, \dots, \mu_\ell$, then there exists an isogeny $X \to Y$ such that Y is minimal and $\mathcal{Y}_{\overline{k}}$ is partially completely slope bi-divisible with respect to $\mu_i, \dots, \mu_\ell$.

Set $\mu = \mu_i$ and write $v_\mu = (s, r)$. Let $\underline{\mathcal{G}}$ be the fppf sheaf obtained as the sheafification of the functor sending an R-algebra A to

$$\mathrm{Im}(\mathrm{Ver}^s : \mathcal{X}^{(p^s)}[p^{s-r}](A) \to \mathcal{X}[p^{s-r}](A)).$$

For an R-algebra S let $\underline{\mathcal{G}}_S$ be the functor obtained by restricting $\underline{\mathcal{G}}$ to S-algebras. Note that $\underline{\mathcal{G}}_k$ (resp. $\underline{\mathcal{G}}_K$) is represented by a finite group scheme $\mathcal{G}_k$ (resp. $\mathcal{G}_K$). We have seen in Lemma 3.14 that $\mathcal{G}_k$ (resp. $\mathcal{G}_K$) is the kernel of $\theta_\mu : \mathcal{X}_k \to \Psi_\mu(\mathcal{X}_k)$ (resp. $\theta_\mu : \mathcal{X}_K \to \Psi_\mu(\mathcal{X}_K)$). Set

$$\mathcal{H} := \mathrm{Ker}(\mathrm{Ver}^s : \mathcal{X}^{(p^s)}[p^{s-r}] \to \mathcal{X}[p^{s-r}]).$$

By the upper-semicontinuity for the structure sheaf of $\mathcal{H}$, we have

$$\mathrm{rk}\, \mathcal{G}_k \le \mathrm{rk}\, \mathcal{G}_K. \tag{21}$$

We claim that $\operatorname{rk}\mathcal{G}_k = \operatorname{rk}\mathcal{G}_K$ if $(\mathcal{X}_k)_i$ in $\mathcal{X}_k$ is not slope bi-divisible with respect to μ. By Proposition 3.17 for $\mathcal{X}_K$, we have

$$\log_p \operatorname{rk}\mathcal{G}_K = \langle v_\mu, \alpha_\mu(\xi)\rangle. \tag{22}$$

Also by Proposition 3.17 again, we get

$$\log_p \operatorname{rk}\mathcal{G}_k \geq \langle v_\mu, \alpha_\mu(\zeta)\rangle, \tag{23}$$

where the equality holds if and only if $(\mathcal{X}_k)_i$ in $\mathcal{X}_k$ is slope bi-divisible with respect to μ. By our assumption, the difference of the right hand sides of (22) and (23) is at most one:

$$\langle v_\mu, \alpha_\mu(\xi) - \alpha_\mu(\zeta)\rangle \leq 1. \tag{24}$$

Clearly (21) $\sim$ (24) imply the claim.

If $\operatorname{rk}\mathcal{G}_k = \operatorname{rk}\mathcal{G}_K$, then $\mathcal{H}$ is flat over R, whence $\underline{\mathcal{G}}$ is represented by a finite flat group scheme $\mathcal{G}$ which is isomorphic to the quotient $\mathcal{X}^{(p^s)}[p^{s-r}]/\mathcal{H}$ (cf. [1, Exp. V]). Putting $\Psi_\mu(\mathcal{X}) = \mathcal{X}/\mathcal{G}$, we have the canonical isogeny $\mathcal{X} \to \Psi_\mu(\mathcal{X})$. Note that $\Psi_\mu(\mathcal{X})_k = \Psi_\mu(\mathcal{X}_k)$ and $\Psi_\mu(\mathcal{X})_K = \Psi_\mu(\mathcal{X}_K)$.

This argument can be applied to $\Psi_\mu(\mathcal{X})$ if $\Psi_\mu(\mathcal{X}_k)_i$ in $\Psi_\mu(\mathcal{X}_k)$ is not slope bi-divisible with respect to μ. Repeating this argument, we have the sequence of isogenies

$$\mathcal{X} \to \Psi_\mu(\mathcal{X}) \to \cdots \to \Psi_\mu^e(\mathcal{X}),$$

where e is the smallest non-negative integer such that $\Psi_\mu^e(\mathcal{X}_k)_i$ in $\Psi_\mu^e(\mathcal{X}_k)$ is slope bi-divisible with respect to μ. Here we used Lemma 3.21 for the existence of e. This sequence is obtained by the flat extension of

$$X \to \Psi_\mu(X) \to \cdots \to \Psi_\mu^e(X),$$

where all $\Psi_\mu^i(X)$ are minimal. Let $X \to Y$ be the isogeny $X \to \Psi_\mu^e(X)$. Then its flat extension $\mathcal{X} \to \mathcal{Y}$ coincides with $\mathcal{X} \to \Psi_\mu^e(\mathcal{X})$. It follows from Proposition 3.22 that $\mathcal{Y}_{\overline{k}}$ is partially completely slope bi-divisible with respect to $\mu_i, \ldots, \mu_\ell$. □

We generalize Proposition 4.1 to the case of general (S, D), using the same technique as in [15].

Theorem 4.2. *Let S be an integral noetherian scheme with prime Weil divisor D. Assume that S is regular at the generic point of D. Let $\mathcal{X}$ be an NP-quasi-saturated p-divisible group over (S, D). Let ξ (resp. ζ) be the Newton polygon of $\mathcal{X}_{S\setminus D}$ (resp. $\mathcal{X}_D$). Let $\{\mu_1, \ldots, \mu_\ell\}$ be a subset of Λ_+ containing all slopes of ζ such that $\langle v_{\mu_i}, \alpha_{\mu_i}(\xi) - \alpha_{\mu_i}(\zeta)\rangle \leq 1$. Suppose $\overline{\mu_1} > \cdots > \overline{\mu_\ell}$. Then there is a finite birational morphism $\pi : T \to S$ such that $\mathcal{X}_T$ is isogenous to a p-divisible group $\mathcal{Y}$ over T such that all the geometric fibers over $T \setminus \pi^{-1}(D)$ are minimal and $\mathcal{Y}_{\pi^{-1}D}$ is completely slope bi-divisible with respect to $\mu_1, \ldots, \mu_\ell$.*

Proof. Let η be the generic point of D. Let $R = \mathcal{O}_{S,\eta}$ and $K = \mathrm{frac}(R)$. Set $X = \mathcal{X}_K$. Let $\rho : X \to Y$ be the isogeny over K constructed in Proposition 4.1.

Let G be the kernel of ρ, and let $\overline{G}$ be the scheme-theoretic image of $G \to \mathcal{X}[p^N]$ for sufficient large N. Let V be the largest open subvariety such that $\overline{G}$ is flat over V. Note that V contains the generic point η of D. We have the p-divisible group $\mathcal{Y}' := \mathcal{X}_V/\overline{G}_V$ over V with isogeny

$$\rho' : \quad \mathcal{X}_V \longrightarrow \mathcal{Y}'.$$

Let d be the degree of ρ. We make use of the moduli space $\mathcal{M}$ of isogenies from $\mathcal{X}$ of degree d. This is defined to be the scheme over S representing the following functor $\underline{\mathcal{M}}$ from the category of S-schemes to that of sets. For an S-scheme T, an element of $\underline{\mathcal{M}}(T)$ is the isomorphism class of an isogeny $\mathcal{X}_T \to Z$ of degree d over T, where Z is a p-divisible group over T. It is known that $\underline{\mathcal{M}}$ is represented by a projective scheme $\mathcal{M}$ over S, see [15], 2.3.

Now ρ' defines a morphism $V \to \mathcal{M}$ commuting the diagram

$$\begin{tikzcd} & \mathcal{M} \arrow[d] \\ V \arrow[ur] \arrow[r] & S. \end{tikzcd}$$

Let $\tilde{S}$ be the scheme-theoretic image of V in $\mathcal{M}$. Then we have a morphism $f : \tilde{S} \to S$, which is proper, surjective and birational. The inclusion $\tilde{S} \subset \mathcal{M}$ defines an isogeny $\mathcal{X}_{\tilde{S}} \to \mathcal{Y}''$ over $\tilde{S}$. Since $\mathcal{Y}''$ is minimal over the generic point of $\tilde{S} \setminus f^{-1}(D)$, by Corollary 3.13 $\mathcal{Y}''$ is minimal over $\tilde{S} \setminus f^{-1}(D)$. Also $\mathcal{Y}''_{f^{-1}(D)}$ is completely slope bi-divisible over every generic point, and therefore $\mathcal{Y}''_{f^{-1}(D)}$ is completely slope bi-divisible by Lemma 3.11.

Let

$$\tilde{S} \longrightarrow T \xrightarrow{\ \pi\ } S$$

be the Stein factorization with $f_*\mathcal{O}_{\tilde{S}} = \mathcal{O}_T$. Let $x \in T$ and let $\tilde{S}_x$ be the fiber over x of $\tilde{S} \to T$. By [15], 2.5 Lemma, the image of $\tilde{S}_{\overline{x}} \to \mathcal{M}$ is finite. Since $\tilde{S}_{\overline{x}}$ is connected, the image is a single point of $\mathcal{M}$. From [15], 2.6 Lemma, we have a morphism $T \to \mathcal{M}$. This defines a desired isogeny

$$\mathcal{X}_T \longrightarrow \mathcal{Y}$$

over T. $\square$

The next is the result in the NP-saturated case, from which Corollary 1.1 follows immediately.

Corollary 4.3. *Let S, D be as in Theorem 4.2. Let $\mathcal{X}$ be an NP-saturated p-divisible group over (S, D). Then there is a finite birational morphism $T \to S$ such that $\mathcal{X}_T$ is isogenous to a p-divisible group $\mathcal{Y}$ over T whose geometric fibers are all minimal.*

Proof. Let ξ (resp. ζ) be the Newton polygon of $\mathcal{X}_{S\setminus D}$ (resp. $\mathcal{X}_D$). As in (5) we write

$$\zeta = \varrho +_{\mathrm{NP}} \zeta' \quad \text{and} \quad \xi = \varrho +_{\mathrm{NP}} \xi' \tag{25}$$

so that $\zeta' \prec \xi'$ is saturated and ξ' consists of only two segments. Let a (resp. b) be the smallest (resp. biggest) slope of ξ'.

In order to apply Theorem 4.2 to $\mathcal{X}$, we need to choose $\mu_1, \ldots, \mu_\ell$ as in Theorem 4.2. We will define them as the union of three kinds of subsets of Λ_1, which will be labeled as A, B and C. (For the definition of Λ_1, see the sentence following (8) in §3. Recall that Λ_1 is canonically identified with the set of slopes $\lambda \in \mathbb{Q}$ with $0 \leq \lambda \leq 1$.) First A is the set of slopes of ζ. Let B be the set of $\nu \in \Lambda_1$ such that v_ν or $-v_\nu$ is equal to $\alpha_\lambda(\xi) - \alpha_\lambda(\zeta)$ for some slope λ of ζ'. For each positive slope λ of ρ, we choose a $\nu \in \Lambda_1$ satisfying the following two properties: (i) $\langle v_\nu, v_\lambda \rangle = 1$ and (ii) $\overline{\nu}$ is sufficiently close to λ so that $\overline{\nu}$ is distinct from the slope of any element of $A \cup B$. Let C be the set of such ν's. Let $\mu_1, \ldots, \mu_\ell$ be the union of A, B and C, and arrange them so that $\overline{\mu_1} > \ldots > \overline{\mu_\ell}$. Theorem 4.2 is applicable for these $\mu_1, \ldots, \mu_\ell$. Indeed

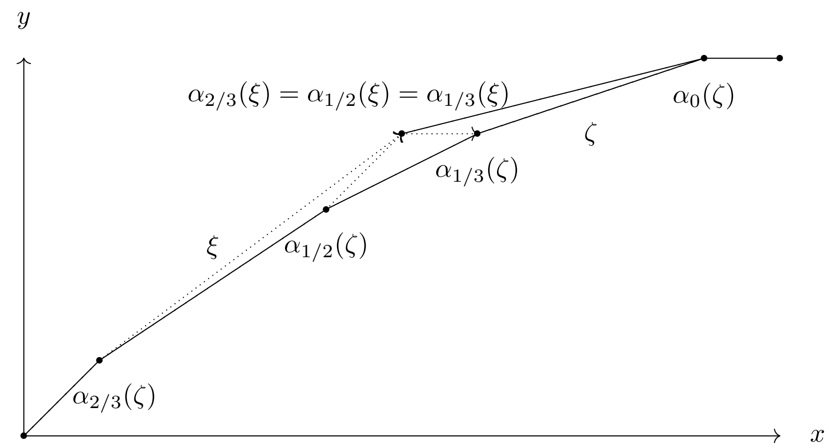
$$\langle v_{\mu_i}, \alpha_{\mu_i}(\xi) - \alpha_{\mu_i}(\zeta) \rangle \leq 1 \tag{26}$$

hold for $i = 1, \ldots, \ell$. For $\mu_i \in A$, this follows from the fact that $\zeta \prec \xi$ is quasi-saturated (Lemma 2.2). For $\mu_i \in B$, the slope of μ_i is outside $[a, b]$; hence the left hand side of (26) is equal to zero. Also for $\mu_i \in C$, the inequality (26) holds.

Let $\mathcal{Y}$ be the p-divisible group obtained by Theorem 4.2. Let s be any geometric point of $\pi^{-1}(D)$. Let λ be any slope of ζ. Let Z_λ be the non-zero $\mathrm{Gr}_i(\mathcal{Y}_s)$ of slope λ. Since $\mathcal{Y}_s$ is completely slope divisible, Z_λ is slope divisible with respect to λ. If λ is zero, then Z_λ is étale and therefore $Z_\lambda \simeq H_{1,0}$, whence this is minimal. If $\lambda > 0$, then there exists $\nu \in \{\mu_1, \ldots, \mu_\ell, \mu_1^*, \ldots, \mu_\ell^*\}$ such that $\langle v_\nu, v_\lambda \rangle = 1$, and Z_λ is slope divisible with respect to ν. It follows from Proposition 3.12 that Z_λ is minimal. Thus every $\mathrm{Gr}_i(\mathcal{Y}_s)$ is minimal, and therefore so is $\mathcal{Y}_s$. $\square$

Example 4.4. For the case of Example 2.3, we illustrate the subsets A, B and C of Λ_1 which appeared in the proof of Corollary 4.3. The saturated pair of Newton polygons is: $\xi = (0,1) +_{\mathrm{NP}} (1,3) +_{\mathrm{NP}} (3,1) +_{\mathrm{NP}} (1,0)$ and $\zeta = (0,1) +_{\mathrm{NP}} (1,2) +_{\mathrm{NP}} (1,1) +_{\mathrm{NP}} (2,1) +_{\mathrm{NP}} (1,0)$. We use the identification of Λ_1 with the set of slopes. First, as A is the set of slopes of ζ, we have

$A = \{1,\ 2/3,\ 1/2,\ 1/3,\ 0\}$. In the picture below, the dotted arrows correspond to the elements of B. So $B = \{1,\ 3/4,\ 0\}$.



Finally, $n/(n+1)$ for any sufficiently large n can be an element of C. If we choose $4/5$, then $C = \{4/5\}$. Thus the union of A, B and C is

$$\{1,\ 4/5,\ 3/4,\ 2/3,\ 1/2,\ 1/3,\ 0\}.$$

5. Application: the configuration of minimal p-kernel types

Recall [2, Corollary 3.2] that the central streams [11, 3.10] in the moduli space of principally polarized abelian varieties are configurated as given by the partial ordering on symmetric Newton polygons. As an application of Corollary 1.1 we shall show its unpolarized analogue (Corollary 5.1), with a geometrical proof, whereas [2] uses a combinatorial method.

Let h be a natural number. Let c, d be non-negative integers with $c+d = h$. Let W be the Weyl group of GL_h. Let $\Delta = \{\alpha_1, \ldots, \alpha_{h-1}\}$ be the set of simple roots as usual. Let s_i be the simple reflection associated to α_i. Set $I = \Delta \backslash \{\alpha_c\}$. Let W_I be the subgroup of W generated by s_i with $\alpha_i \in I$. Let ${}^IW$ be the set of the minimal-length representatives of $W_I \backslash W$.

Let k be an algebraically closed field. Recall the classification theory of truncated Barsotti-Tate groups of level one (BT_1's) over k found by Kraft [8] and rediscovered by Oort and reproved and formulated as follows by Moonen-Wedhorn [9]. It says that there exists a canonical bijection from ${}^IW$ to the set of isomorphism classes of BT_1's over k of codimension c and of dimension d.

We use F-zips, which in this paper mean those with support contained in $\{0, 1\}$ in the terminology of [9]. Let S be a scheme in characteristic $p > 0$. An *F-zip* over S is a quintuple $(N, C, D, \varphi, \dot\varphi)$ consisting of locally free $\mathcal{O}_S$-module N and $\mathcal{O}_S$-submodules C, D of N which are locally direct summands of N with $\mathcal{O}_S$-linear isomorphisms $\varphi : (N/C)^{(p)} \to D$ and $\dot\varphi : C^{(p)} \to N/D$. Let G be a

BT_1 over k. To G we associate an F-zip $(\mathbb{D}(G), V\,\mathbb{D}(G), F\,\mathbb{D}(G), F, V^{-1})$. This gives a canonical bijection from the set of BT_1's over k and the set of F-zips over k.

Let $w_\xi \in {}^IW$ denote the p-kernel type of the minimal p-divisible group $H(\xi)_k$ of Newton polygon ξ. For $v, w \in {}^IW$ we say $v \subset w$ if there exists an F-zip over a discrete valuation ring of which the generic fiber (resp. the special fiber) is of type w (resp. of type v). It follows from [16, Theorem 12.17] that $\subset$ is a partial ordering on ${}^IW$ and this coincides with the partial ordering introduced and investigated by He [5].

Corollary 5.1. *$w_\zeta \subset w_\xi$ if and only if $\zeta \preceq \xi$.*

Proof. For the "if"-part, since $\subset$ is a partial ordering, it is enough to show the case that $\zeta \prec \xi$ is saturated. Applying Corollary 1.1 to a family with saturated $\zeta \prec \xi$ and with a-number ≤ 1, constructed in [12], (3.2), we have $w_\zeta \subset w_\xi$.

Suppose $w_\zeta \subset w_\xi$. There exists an F-zip $\mathcal{N}$ over a discrete valuation ring R with algebraically closed residue field whose special fiber is of type w_ζ and whose generic fiber is of type w_ξ. Then there exists a display $\mathcal{M}$ over R such that $\mathcal{M}/I_R\mathcal{M}$ is isomorphic to $\mathcal{N}$, see [4, Lemma 4.1]. By [10], the special fiber (resp. the generic fiber) of $\mathcal{M}$ is minimal of Newton polygon ζ (resp. ξ). By Grothendieck-Katz [7, Th. 2.3.1 on p. 143], we have $\zeta \preceq \xi$. □

Combining this with [4, Theorem 1.1], one can get the unpolarized analogue of Oort's conjecture [11, 6.9]. The original conjecture was proved in [3] and [13], also see [14] for a generalization to some Shimura varieties.

Corollary 5.2. *If there exists a p-divisible group with Newton polygon ξ and p-kernel type w, then we have $w_\xi \subset w$.*

Proof. Let $\xi(w)$ be the supremum of Newton polygons of p-divisible groups with p-kernel type w. We have $\xi \preceq \xi(w)$. From Corollary 5.1 it follows that $w_\xi \subset w_{\xi(w)}$. Recall [4, Theorem 1.1], which says that $\xi(w)$ is the maximal one among Newton polygons η with $w_\eta \subset w$. In particular we have $w_{\xi(w)} \subset w$. □

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