Ekedahl-Oort strata contained in the supersingular locus and Deligne-Lusztig varieties

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Dedicated to Professor Tomoyoshi Ibukiyama on his 60th birthday

Abstract

We study the moduli space of principally polarized abelian varieties over fields of positive characteristic. In this paper we describe certain unions of Ekedahl-Oort strata contained in the supersingular locus in terms of Deligne-Lusztig varieties. As a corollary we show that each Ekedahl-Oort stratum contained in the supersingular locus is reducible except possibly for small p.

1 Introduction

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties over fields of characteristic $p > 0$. The moduli space $\mathcal{A}_g$ has the Newton polygon stratification, which is defined by isogeny types of p-divisible groups. Ekedahl and Oort introduced another new stratification on $\mathcal{A}_g$ in [27], which is now called the Ekedahl-Oort stratification: two principally polarized abelian varieties A and B are in the same stratum if and only if there exists an isomorphism between their p-kernels $A[p]$ and $B[p]$ over an algebraically closed field. The isomorphism classes of such p-kernels are classified by elements of a subset ${}^I W_g$ of the Weyl group W_g of the symplectic group Sp_{2g}. We write $\mathcal{S}_w$ for the Ekedahl-Oort stratum related to $w \in {}^I W_g$.

In the usual way we identify W_g with

$$\{ w \in \mathrm{Aut}\{1, \ldots, 2g\} \mid w(i) + w(2g+1-i) = 2g+1 \}. \tag{1}$$

Let $\{s_1, \ldots, s_g\}$ be the set of simple reflections, where $s_i = (i, i+1) \cdot (2g-i, 2g+1-i)$ for $i < g$ and $s_g = (g, g+1)$. Let $I = \{s_1, \ldots, s_{g-1}\}$ and let $W_{g,I}$ be the subgroup of W_g generated by elements of I. We denote by ${}^I W_g$ the set of $(I, \emptyset)$-reduced elements of W_g (cf. [3], Chap. IV, Ex. §1, 3), which is a set of representatives of $W_{g,I} \backslash W_g$. We also write $\overline{W}_g$ for $W_{g,I} \backslash W_g / W_{g,I}$. Note ${}^I W_g$ is explicitly given by

$${}^I W_g = \left\{ \ w \in W_g \ \middle| \ w^{-1}(1) < \cdots < w^{-1}(g) \right\}. \tag{2}$$

For $c \leq g$ let

$$ {}^I W_g^{[c]} = \{ w \in {}^I W_g \mid w(i) = i, \ \forall\, i \leq g - c \}, \tag{3} $$

and set ${}^I W_g^{(c)} = {}^I W_g^{[c]} - {}^I W_g^{[c-1]}$ for $c > 0$ and ${}^I W_g^{(0)} = {}^I W_g^{[0]} = \{\mathrm{id}\}$. We define a map

$$ \mathfrak{r} : \quad {}^I W_g^{(c)} \longrightarrow \overline{W}_c \tag{4} $$

by sending w to the class of $v \in W_c$ determined by $v(i) = w(g - c + i) - (g - c)$ for all $1 \leq i \leq c$. We denote by $\overline{W}'_c$ the image of $\mathfrak{r}$.

Assume $c \leq \lfloor g/2 \rfloor$. For $w' \in \overline{W}'_c$, we shall investigate the union $\mathcal{J}_{w'}$ of the Ekedahl-Oort strata $\mathcal{S}_w$ with $w \in {}^I W_g^{(c)}$ and $\mathfrak{r}(w) = w'$:

$$ \mathcal{J}_{w'} = \bigcup_{\mathfrak{r}(w) = w'} \mathcal{S}_w. \tag{5} $$

For each c, we fix once and for all a symplectic vector space $(L_0, \langle\ ,\ \rangle)$ over $\mathbb{F}_{p^2}$ of dimension $2c$ and a maximal totally isotropic subspace C_0 over $\mathbb{F}_{p^2}$ of L_0. Let $\mathrm{Sp}(L_0)$ denote the symplectic group over $\mathbb{F}_{p^2}$ associated to $(L_0, \langle\ ,\ \rangle)$. Let P_0 be the parabolic subgroup of $\mathrm{Sp}(L_0)$ stabilizing C_0. Let X be the flag variety $\mathrm{Sp}(L_0)/\mathrm{P}_0$ over $\mathbb{F}_{p^2}$. For $w' \in \overline{W}_c$, let $\mathrm{X}(w')$ be the Deligne-Lusztig variety in X related to w'. We shall review the definition of Deligne-Lusztig varieties in §2.6.

Main theorem. *Assume $c \leq \lfloor g/2 \rfloor$. For each $w' \in \overline{W}'_c$, there exists a finite surjective morphism*

$$ \mathrm{G}(\mathbb{Q}) \backslash \mathrm{X}(w') \times \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K} \longrightarrow \mathcal{J}_{w'} $$

over $\mathbb{F}_{p^2}$, which is bijective on geometric points, see §3.2 for the definition of the quaternion unitary group G over $\mathbb{Z}$ and the open compact subgroup K of $\mathrm{G}(\mathbb{A}^\infty)$ and see §3.4 for the $\mathrm{G}(\mathbb{Q})$-action on $\mathrm{X}(w')$.

In this paper we shall prove this theorem in the case of $g \geq 2$. If $g = 1$, then ${}^I W_1^{(0)} = \{\mathrm{id}\}$ and $\mathcal{S}_{\mathrm{id}}$ consists of the supersingular elliptic curves, see Deuring [7] and Igusa [14] for this case. The case of $g = 2$ has been studied by Ibukiyama-Katsura-Oort [13] and Katsura-Oort [16]. In the case of unitary Shimura varieties, Vollaard [30] has dealt with the decompositions of some basic loci into Deligne-Lusztig varieties.

The main theorem above seems to have been refined by Hoeve, see [12] where he described individual Ekedahl-Oort strata contained in the supersingular locus in terms of "fine" Deligne-Lusztig varieties. We also mention the paper [31] by Vollaard and Wedhorn, in which they proved an analogous result in the case of unitary Shimura varieties. Deligne-Lusztig varieties also appear in the paper [9] by Görtz and Yu, where they studied supersingular Kottwitz-Rapoport strata.

Thanks to Bonnafé and Rouquier [1], we have a corollary to the main theorem (see §3.5 for further details):

Corollary. *For any $w' \in \overline{W}'_c$ with $c \leq \lfloor g/2 \rfloor$, the number of irreducible (connected) components of $\mathcal{J}_{w'}$ equals the class number $H_{g,c} = \sharp\, \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}$.*

Oort conjectured that (i) $\mathcal{S}_w$ is irreducible if $\mathcal{S}_w$ is not contained in the supersingular locus and (ii) $\mathcal{S}_w$ is reducible for sufficiently large p otherwise. In Remark 2.5.7 we shall see that $\mathcal{S}_w$ is contained in the supersingular locus if and only if $w \in {}^I W_g^{(c)}$ with $c \leq \lfloor g/2 \rfloor$. Ekedahl and van der Geer proved in [8], Theorem 11.5 that $\mathcal{S}_w$ is irreducible for every $w \in {}^I W_g^{(c)}$ with $c > \lfloor g/2 \rfloor$; thus (i) was proved. In the last section we shall confirm (ii) by showing $\lim_{p\to\infty} H_{g,c} = \infty$.

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2 Preliminaries

We recall some basic definitions and facts on the Dieudonné theory, abelian varieties, the Ekedahl-Oort stratification and Deligne-Lusztig varieties.

2.1 The Dieudonné theory over a perfect field

We fix once and for all a rational prime p. Let k be a perfect field of characteristic p. We denote by $W(k)$ the ring of Witt vectors with coefficients in k. We define a ring $\mathbb{E} = \mathbb{E}_k$ by the p-adic completion of

$$W(k)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \mathcal{F}a - {}^\sigma a\mathcal{F}, \mathcal{V}^\sigma a - a\mathcal{V}, \forall a \in W(k)). \quad (6)$$

Here σ is the Frobenius map on $W(k)$. A *Dieudonné module* over k is a left $\mathbb{E}$-module M which is finitely generated as a $W(k)$-module. If M is also free as a $W(k)$-module, we call M a *free* Dieudonné module.

The covariant Dieudonné theory says that there is a canonical categorical equivalence $\mathbb{D}$ from the category of p-torsion finite commutative group schemes (resp. p-divisible groups) over k to the category of Dieudonné modules over k which are of finite length (resp. free as $W(k)$-modules). We write F and V for "Frobenius" and "Verschiebung" on commutative group schemes. The covariant Dieudonné functor $\mathbb{D}$ satisfies $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$. The *dual*

Dieudonné module M^t is the $W(k)$-module $\mathrm{Hom}_{W(k)}(M, W(k))$ with $\mathcal{F}$ and $\mathcal{V}$-operators defined by $(\mathcal{F}f)(x) = {}^{\sigma}f(\mathcal{V}x)$ and $(\mathcal{V}f)(x) = {}^{\sigma^{-1}}f(\mathcal{F}x)$ for any $f \in \mathrm{Hom}_{W(k)}(M, W(k))$ and $x \in M$. For an abelian variety Y over k, we have a free Dieudonné module $\mathbb{D}(Y) := \mathbb{D}(Y[p^\infty])$. The Dieudonné module $\mathbb{D}(Y^t)$ of the dual abelian variety Y^t is canonically isomorphic to $\mathbb{D}(Y)^t$.

Let M be a free Dieudonné module over k. A *quasi-polarization* on M is a non-degenerate $W(k)$-bilinear alternating form

$$\langle \ , \ \rangle : M \otimes_{W(k)} M \to W(k) \tag{7}$$

satisfying $\langle \mathcal{F}x, y\rangle = {}^{\sigma}\langle x, \mathcal{V}y\rangle$. A quasi-polarization is called *principal* if it is a perfect pairing. By [23, p.101], a polarization λ on an abelian variety Y induces a quasi-polarization $\langle \ , \ \rangle_\lambda$ on $\mathbb{D}(Y)$; moreover λ is p-principal if and only if $\langle \ , \ \rangle_\lambda$ is principal.

Two free Dieudonné modules M and N are said to be *isogenous* if there exists an injective $\mathbb{E}$-homomorphism from M to N with torsion cokernel. A free Dieudonné module M is called *supersingular* (resp. *superspecial*) if M is isogenous (resp. isomorphic) to $\mathbb{E}_{1,1}^{\oplus g}$ over an algebraically closed field, where $\mathbb{E}_{1,1} := \mathbb{E}/\mathbb{E}(\mathcal{F} - \mathcal{V})$. For a free Dieudonné module M, the a-number $a(M)$ of M is defined to be $\dim_k M/(\mathcal{F}, \mathcal{V})M$. We have $a(M) = g$ if and only if M is superspecial ([18], p. 32).

An abelian variety Y over k is said to be *supersingular* (resp. *superspecial*) if the Dieudonné module $\mathbb{D}(Y)$ is supersingular (resp. superspecial). An abelian variety Y is supersingular if and only if there is an isogeny from E^g to Y over an algebraically closed field (cf. [25], Theorem 4.2 and [24], Theorem 6.2), where E is a supersingular elliptic curve. For $g \geq 2$, an abelian variety Y is superspecial if and only if there is an isomorphism between Y and E^g over an algebraically closed field (this condition does not depend on the choice of E, see [24], Theorem 6.2 and [29], Theorem 3.5).

2.2 α-groups

Let S be any scheme. A locally free finite group scheme G over S is called an *α-group* if both $F_{G/S} : G \to G^{(p)}$ and $V_{G/S} : G^{(p)} \to G$ are zero.

For α-groups, the covariant Dieudonné functor is extended as follows: there is an equivalence of categories from the category of α-groups over S to the category of locally free sheaves of finite rank on S, which is defined by composing the Cartier dual functor and the anti-equivalence obtained in [15], Proposition 2.2. Also see [17], Corollary 2.6 or [18], 2.4. Let G be an α-group over S and let $\mathcal{L}$ be the associated locally free sheaf on S. Then the locally free sheaf associated to the Cartier dual G^D is isomorphic to the locally free sheaf $\mathcal{H}om(\mathcal{L}, \mathcal{O}_S)$.

2.3 Descent of polarizations

Let (Z, μ) be a polarized abelian scheme over S and set $G := \mathrm{Ker}\,\mu$. Note that μ induces an isomorphism $\imath : G \to G^D$ (cf. [18], 3.7). Assume G is an α-group.

Associated to G, we have a locally free sheaf $\mathcal{L}$. Let $\jmath$ be the isomorphism $\mathcal{L} \to \mathcal{H}om(\mathcal{L}, \mathcal{O}_S)$ induced by $\imath$, which gives an alternating perfect pairing on $\mathcal{L}$. Let $\rho : Z \to Y$ be an isogeny of abelian schemes and set $H = \operatorname{Ker} \rho$. Assume $H \subset G$. Then H is also an α-group. As shown in the proof of [18], 3.7, Lemma, μ descends to a polarization on Y if and only if the composition $H \subset G \xrightarrow{\imath} G^D \to H^D$ is zero. Let $\mathcal{I}$ be the locally free subsheaf of $\mathcal{L}$ associated to H.

Lemma 2.3.1. *μ descends to a polarization on Y if and only if $\mathcal{I}$ is totally isotropic in $\mathcal{L}$.*

Proof. Paraphrasing the condition that $H \subset G \xrightarrow{\imath} G^D \to H^D$ is zero, we have that the composition $\mathcal{I} \subset \mathcal{L} \xrightarrow{\jmath} \mathcal{H}om(\mathcal{L}, \mathcal{O}_S) \to \mathcal{H}om(\mathcal{I}, \mathcal{O}_S)$ is zero. This is nothing but the condition that $\mathcal{I}$ is totally isotropic in $\mathcal{L}$. $\square$

2.4 Minimal isogenies

Recall [17], Lemma 1.3:

Lemma 2.4.1. *Let k be a perfect field of characteristic p. For a supersingular Dieudonné module over k, there is a biggest superspecial Dieudonné module $S_0(M)$ contained in M, and dually there exists a smallest superspecial Dieudonné module $S^0(M)$ in $M \otimes \operatorname{frac} W(k)$ containing M, where $\operatorname{frac}(W(k))$ denotes the fractional field of $W(k)$.*

Let K be an arbitrary field of characteristic p. We denote by K^{pf} the perfect hull of K. For supersingular p-divisible groups $\mathcal{G}$ and $\mathcal{H}$ over K, an isogeny $\mathcal{H} \to \mathcal{G}$ over K is called *minimal* if the induced isogeny $\mathbb{D}(\mathcal{H} \otimes K^{\mathrm{pf}}) \to \mathbb{D}(\mathcal{G} \otimes K^{\mathrm{pf}})$ is isomorphic to the inclusion $S_0(M) \to M$ with $M = \mathbb{D}(\mathcal{G} \otimes K^{\mathrm{pf}})$. We also recall [18], 1.8:

Lemma 2.4.2. *For a supersingular abelian variety Y over K, there exists a superspecial abelian variety Z over K and a K-isogeny $\rho : Z \to Y$ such that for any superspecial abelian variety Z' over K and any K-isogeny $\rho' : Z' \to Y$, there is a unique K-isogeny $\phi : Z' \to Z$ such that $\rho' = \rho \circ \phi$. (The isogeny ρ is also called a minimal isogeny.)*

2.5 The Ekedahl-Oort stratification

The main reference for the EO-stratification is [27]. For a formulation in terms of Weyl groups, see [8], [19] and [20].

Definition 2.5.1. (1) A finite locally free commutative group scheme G over $\mathbb{F}_p$-scheme S is said to be a BT_1 over S if it is annihilated by p and $\operatorname{Im}(V : G^{(p)} \to G) = \operatorname{Ker}(F : G \to G^{(p)})$.

(2) Assume k is perfect. Let G be a BT_1 over k. A *symmetry* of G is an isomorphism from G to its Cartier dual G^D. A symmetry $\imath$ is called a *polarization* if the bilinear form $\langle\ ,\ \rangle : \mathbb{D}(G) \otimes_k \mathbb{D}(G) \to k$ induced by $\imath$ is alternating. Such a pair $(G, \imath)$ is called a *polarized* BT_1.

Recall the classification of polarized BT_1's.

Theorem 2.5.2. *Let k be an algebraically closed field. There is a canonical bijection*

$$\mathcal{E}: \quad \{\textit{polarized } \mathrm{BT}_1 \textit{ over } k\} / \simeq \xrightarrow{\sim} {}^I W_g.$$

Remark 2.5.3. This classification was obtained by Oort [27], (9.4) and Moonen-Wedhorn [20], (5.4), also see Moonen [19]. Instead of ${}^I W_g$, Oort used the set of elementary sequences (or symmetric final sequences), see below for the definition of them. The above formulation in terms of Weyl groups is due to Moonen-Wedhorn.

A *symmetric final sequence* of length $2g$ is a map

$$\psi: \quad \{0, \dots, 2g\} \longrightarrow \{0, \dots, g\}$$

such that $\psi(i-1) \leq \psi(i) \leq \psi(i-1)+1$ for $1 \leq i \leq 2g$ with $\psi(0)=0$ and $\psi(2g-i) = g - i + \psi(i)$. To each element w of ${}^I W_g$, we associate a symmetric final sequence ψ_w defined by

$$\psi_w(i) = \sharp\{a \in \{1, \dots, i\} \mid w(a) > g\}. \tag{8}$$

This correspondence gives a bijection from ${}^I W_g$ to the set of symmetric final sequences of length $2g$. An *elementary sequence* of length g is the restriction of a symmetric final sequence of length $2g$ to $\{1, \dots, g\}$. Clearly to give an elementary sequence of length g is equivalent to giving a symmetric final sequence of length $2g$.

Lemma 2.5.4. *For $w \in {}^I W_g$, we have $w \in {}^I W_g^{[c]}$ if and only if $\psi_w(g-c) = 0$.*

Proof. If $w \in {}^I W_g^{[c]}$, then $w(i) = i$ for $i \leq g-c$ by definition; hence we obtain $\psi_w(g-c) = 0$ by (8). Conversely assume $\psi_w(g-c)=0$. Then we have $w(i) \leq g$ for all $i \leq g-c$. Since $w \in {}^I W_g$, i.e., $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g)$, we have $w(i) = i$ for all $i \leq g-c$. □

Let us recall the definition of the map $\mathcal{E}$ in Theorem 2.5.2. Let G be a polarized BT_1 over an algebraically closed field k and let N be the Dieudonné module of G. We define an operator $\mathcal{V}^{-1}$ on the set of Dieudonné submodules N' of N by

$$\mathcal{V}^{-1}N' := \mathcal{V}^{-1}(N' \cap \mathcal{V}(N)), \tag{9}$$

and inductively we define a Dieudonné submodule sN' of N for any word s of $\mathcal{F}$ and $\mathcal{V}^{-1}$. It was shown in [27], (2.4) that there exists a unique $w \in {}^I W_g$ satisfying $\mathrm{rk}(\mathcal{F}sN) = \psi_w(\mathrm{rk}\, sN)$ and $\mathrm{rk}(\mathcal{V}^{-1}sN) = g + \mathrm{rk}\, sN - \psi_w(\mathrm{rk}\, sN)$ for any word s. Then we define $\mathcal{E}(G) = w$.

Let G be a polarized BT_1 over an algebraically closed field k. Let $w = \mathcal{E}(G)$ and put $\psi := \psi_w$. By [27], (9.4), we can express $N = \mathbb{D}(G)$ as follows:

$$N = \bigoplus_{i=1}^{2g} k b_i \tag{10}$$

with the operators $\mathcal{F}$ and $\mathcal{V}$ defined by

$$\mathcal{F}(b_i) := \begin{cases} b_{\psi(i)} & \text{if} \quad w(i) > g, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

$$\mathcal{V}(b_j) := \begin{cases} b_i & \text{if} \quad j = g + i - \psi(i) \text{ with } w(i) \le g \text{ and } w(j) \le g, \\ -b_i & \text{if} \quad j = g + i - \psi(i) \text{ with } w(i) \le g \text{ and } w(j) > g, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and the polarization $\langle\ ,\ \rangle$ defined by

$$\langle b_i, b_{2g+1-j} \rangle = \begin{cases} 1 & \text{if} \quad i = j \quad \text{and} \quad w(i) > g, \\ -1 & \text{if} \quad i = j \quad \text{and} \quad w(i) \le g, \\ 0 & \text{if} \quad i \ne j. \end{cases} \tag{13}$$

We shall use the following:

Lemma 2.5.5. (1) *$\mathcal{F}N$ has a basis $\{b_1, \ldots, b_g\}$ and*

(2) *$\mathcal{V}N$ has a basis $\{b_{w^{-1}(1)}, \ldots, b_{w^{-1}(g)}\}$.*

Proof. Obvious from (11) and (12). □

For $w \in {}^I W_g$, the Ekedahl-Oort stratum $\mathcal{S}_w$ is defined to be the subset of $\mathcal{A}_g$ consisting of points $y \in \mathcal{A}_g$ where y comes over some field from a principally polarized abelian variety A_y such that $\mathcal{E}(A_y[p]) = w$, see [27], (5.11). As shown in [27], (3.2), $\mathcal{S}_w$ is locally closed in $\mathcal{A}_g$; we consider this as a locally closed subscheme by giving it the reduced induced scheme structure.

Recall the result of Ekedahl and van der Geer:

Theorem 2.5.6 ([8], Theorem 11.5)**.** *Assume $w \in {}^I W^{(c)}$ with $c > \lfloor g/2 \rfloor$. Then $\mathcal{S}_w$ is irreducible.*

Remark 2.5.7. Let $w \in {}^I W^{(c)}$. By Lemma 2.5.4 the condition $c \le \lfloor g/2 \rfloor$ is equivalent to $\psi_w(\lfloor (g+1)/2 \rfloor) = 0$. This is also equivalent to that $\mathcal{S}_w$ is contained in the supersingular locus, see [5], (3.7), Step 2 and [11]. Also see Proposition 3.1.5 below.

2.6 Flag varieties and Deligne-Lusztig varieties

We recall the precise definitions of flag varieties and Deligne-Lusztig varieties used in this paper.

Let $(L_0, \langle\ ,\ \rangle)$ and P_0 be as in §1. Let $\mathbb{F}$ be a finite field containing $\mathbb{F}_{p^2}$. Let $\mathfrak{X}$ be the functor from the category of $\mathbb{F}$-schemes to the category of sets, sending S to the set of totally isotropic locally free subsheaves of rank c of $\pi^* L_0$, where $\pi : S \to \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(\mathbb{F}_{p^2})$. It is known that $\mathfrak{X}$ is representable. We define the flag variety $\mathrm{X}(= \mathrm{X}_\mathbb{F})$ to be the scheme representing $\mathfrak{X}$. It is known that X is regular and projective. For any algebraically closed field k over $\mathbb{F}$, there exists a canonical bijection from $\mathrm{X}(k)$ to the set of parabolic subgroups of the form ${}^h(\mathrm{P}_0 \otimes k) := h(\mathrm{P}_0 \otimes k)h^{-1}$ for some $h \in \mathrm{Sp}(L_0)(k)$, by sending a totally isotopic subspace of $L_0 \otimes k$ to its stabilizer group. Note that for $h \in \mathrm{Sp}(L_0)(k)$ we have ${}^h(\mathrm{P}_0 \otimes k) = \mathrm{P}_0 \otimes k$ if and only if $h \in \mathrm{P}_0(k)$.

Let w' be an element of $\overline{W}_c$. Let K be a field containing $\mathbb{F}$. Let $x, y \in \mathrm{X}(K)$ and let P, Q be the parabolic subgroups of $\mathrm{Sp}(L_0 \otimes K)$ stabilizing x and y respectively. We say x and y are in relative position w' if over an algebraic closure k of K there exists an $h \in \mathrm{Sp}(L_0)(k)$ such that we have ${}^h(\mathrm{P} \otimes k) = \mathrm{P}_0 \otimes k$ and ${}^h(\mathrm{Q} \otimes k) = {}^{v'}(\mathrm{P}_0 \otimes k)$ for a lift $v' \in W_c$ of w'. We define $\mathrm{X}(w')(= \mathrm{X}(w')_{\mathbb{F}})$ to be the subset of X consisting of points $x \in \mathrm{X}$ such that x and $\mathrm{Fr}(x)$ are in relative position w', where Fr is the square of the absolute Frobenius on X. It is known that $\mathrm{X}(w')$ is locally closed in X; we consider this as a locally closed subscheme of X by giving it the reduced induced scheme structure. We call $\mathrm{X}(w')$ the Deligne-Lusztig variety related to w'. By the same argument as in [6], 1.3, one can check that $\mathrm{X}(w')$ is regular.

3 Proof of the main results

The substantial part of the proof is §3.1. Here we associate a "flag" (i.e., a maximal totally isotropic subspace in a symplectic vector space) to each principally quasi-polarized Dieudonné module M over an algebraically closed field *under the condition* $c \leq \lfloor g/2 \rfloor$, and describe the condition $\mathfrak{r}(\mathcal{E}(M/pM)) = w'$ in terms of the flag. In §3.2 we review a classification of polarizations on the superspecial abelian varieties by making use of some arithmetic of quaternion unitary groups. In §3.3 we introduce the moduli space $\mathcal{T}_{\mu,\theta}(w')$ of certain isogenies of polarized supersingular abelian varieties and describe $\mathcal{J}_{w',n}$ in terms of $\mathcal{T}_{\mu,\theta}(w')$; moreover by using the result of §3.1 we show that $\mathcal{T}_{\mu,\theta}(w')$ is isomorphic to the Deligne-Lusztig variety $\mathrm{X}(w')$. §3.4 is just a paraphrase of the result of §3.3. In the last subsection §3.5 we enumerate the irreducible components of $\mathcal{J}_{w'}$ and show the reducibility of Ekedahl-Oort strata contained in the supersingular locus.

3.1 Principally quasi-polarized Dieudonné modules with $c \leq \lfloor g/2 \rfloor$

Let k be an algebraically closed field of characteristic p. Let M be a principally quasi-polarized Dieudonné module of genus g over k. Set $N = M/pM$. For any Dieudonné submodule T of M, we write $\overline{T}$ for the $k[\mathcal{F}, \mathcal{V}]$-submodule $T/T \cap pM$ of N. For a $k[\mathcal{F}, \mathcal{V}]$-submodule S of N, we define a Dieudonné submodule $\langle\langle S \rangle\rangle$ of M by

$$\langle\langle S \rangle\rangle = \{x \in M \mid (x \bmod p) \in S\},$$

and let $\mathcal{V}^{-1}S$ be as defined in (9). For an $\mathbb{E}_k$-submodule M' of M we denote by $\mathcal{V}^{-1}M'$ the $\mathbb{E}_k$-module $\{m \in M \otimes_{W(k)} \mathrm{frac}(W(k)) \mid \mathcal{V}m \in M'\}$. Then we have $\langle\langle \mathcal{V}^{-1}S \rangle\rangle = \mathcal{V}^{-1}\langle\langle S \rangle\rangle \cap M$ and $\langle\langle \mathcal{F}S \rangle\rangle = \mathcal{F}\langle\langle S \rangle\rangle + pM$, see [11], Lemma 6.2.

Lemma 3.1.1. *Let S be a $k[\mathcal{F},\mathcal{V}]$-submodule of N such that $\mathcal{V}N \subset S$. Then we have $\langle\langle \mathcal{V}^{-1}\mathcal{F}S\rangle\rangle = \mathcal{V}^{-1}\mathcal{F}\langle\langle S\rangle\rangle \cap M$.*

Proof. We have $\langle\langle \mathcal{V}^{-1}\mathcal{F}S\rangle\rangle = \mathcal{V}^{-1}\langle\langle \mathcal{F}S\rangle\rangle \cap M = \mathcal{V}^{-1}(\mathcal{F}\langle\langle S\rangle\rangle + pM) \cap M$. Clearly $\mathcal{V}N \subset S$ implies $\mathcal{V}M \subset \langle\langle S\rangle\rangle$; hence $pM \subset \mathcal{F}\langle\langle S\rangle\rangle$. Thus we obtain $\langle\langle \mathcal{V}^{-1}\mathcal{F}S\rangle\rangle = \mathcal{V}^{-1}\mathcal{F}\langle\langle S\rangle\rangle \cap M$. □

Lemma 3.1.2. *Assume there exists a $k[\mathcal{F},\mathcal{V}]$-submodule S of N such that $\mathcal{V}N \subset S$ and $\mathcal{V}^{-1}\mathcal{F}S = S$. Then $\langle\langle S\rangle\rangle$ is a superspecial Dieudonné module, and therefore M is supersingular.*

Proof. We have $\langle\langle S\rangle\rangle = \langle\langle \mathcal{V}^{-1}\mathcal{F}S\rangle\rangle = \mathcal{V}^{-1}\mathcal{F}\langle\langle S\rangle\rangle \cap M$ by Lemma 3.1.1. From this, we have $\mathcal{V}\langle\langle S\rangle\rangle \subset \mathcal{F}\langle\langle S\rangle\rangle$; then the a-number $\dim_k \langle\langle S\rangle\rangle/(F,V)\langle\langle S\rangle\rangle$ is equal to g. Hence $\langle\langle S\rangle\rangle$ is superspecial. Then M is supersingular, since $\mathcal{V}M \subset \langle\langle S\rangle\rangle \subset M$. □

From the obvious inclusion $(V^{-1}F)N \subset N$, we have a descending filtration

$$\cdots \subset (\mathcal{V}^{-1}\mathcal{F})^2 N \subset (\mathcal{V}^{-1}\mathcal{F})N \subset N. \tag{14}$$

Since N is of finite length, the filtration is stable. Hence $(\mathcal{V}^{-1}\mathcal{F})^\infty N$ is defined.

Lemma 3.1.3. *Assume that M is supersingular and $\mathcal{V}M \subset S_0(M) \subset M$. Then we have $\overline{S_0(M)} = (\mathcal{V}^{-1}\mathcal{F})^\infty N$.*

Proof. By Lemma 3.1.1, we have $\langle\langle \mathcal{V}^{-1}\mathcal{F}\overline{S_0(M)}\rangle\rangle = \mathcal{V}^{-1}\mathcal{F}S_0(M) \cap M = S_0(M)$; hence $\mathcal{V}^{-1}\mathcal{F}\overline{S_0(M)} = \overline{S_0(M)}$. Applying $(\mathcal{V}^{-1}\mathcal{F})^\infty$ to the both sides of $\overline{S_0(M)} \subset N$, we obtain $\overline{S_0(M)} \subset (\mathcal{V}^{-1}\mathcal{F})^\infty N$.

Note $\mathcal{V}N \subset \overline{S_0(M)} \subset (\mathcal{V}^{-1}\mathcal{F})^\infty N$ and $(\mathcal{V}^{-1}\mathcal{F})(\mathcal{V}^{-1}\mathcal{F})^\infty N = (\mathcal{V}^{-1}\mathcal{F})^\infty N$. Hence $\langle\langle (\mathcal{V}^{-1}\mathcal{F})^\infty N\rangle\rangle$ is superspecial (Lemma 3.1.2). Since $S_0(M)$ is the largest superspecial Dieudonné submodule of M, we have $\langle\langle (\mathcal{V}^{-1}\mathcal{F})^\infty N\rangle\rangle \subset S_0(M)$; hence $(\mathcal{V}^{-1}\mathcal{F})^\infty N \subset \overline{S_0(M)}$. □

Let $w = \mathcal{E}(N)$ and set $\psi = \psi_w$. We choose a basis $\{b_1, \ldots, b_{2g}\}$ of N as satisfying (11), (12) and (13). Let N_i be the subspace of N generated by $b_1, \ldots, b_i$. Then we have a filtration

$$N_0 \subset \cdots \subset N_{2g}. \tag{15}$$

Note $\mathcal{F}N_i = N_{\psi(i)}$ and $\mathcal{V}^{-1}N_i = N_{g+i-\psi(i)}$.

Lemma 3.1.4. *Assume $w \in {}^I W_g^{(c)}$. Then we have*

(1) $\mathcal{V}N \subset N_{2g-c}$;

(2) $\mathcal{V}^{-1}\mathcal{F}N_{2g-c} = N_{2g-c}$ *if* $c \leq \lfloor g/2 \rfloor$.

In particular if $c \leq \lfloor g/2 \rfloor$, then $\langle\langle N_{2g-c}\rangle\rangle$ is superspecial by Lemma 3.1.2.

Proof. (1) It suffices to show that $b_i \notin \mathcal{V}N \Leftarrow i > 2g-c$. Clearly we have the equivalences

$$b_i, \dots, b_{2g} \notin \mathcal{V}N \;\Leftrightarrow\; w(i), \dots, w(2g) > g \;\Leftrightarrow\; w(1), \dots, w(2g+1-i) \leq g.$$

Since $w^{-1}(1) < \cdots < w^{-1}(g)$, the last condition is nothing but $w(j) = j$ for $1 \leq j \leq 2g+1-i$, namely $w \in {}^I W_g^{[2g+1-i]}$. This condition is equivalent to $i > 2g-c$.

(2) By $c \leq g-c$, we have $\psi_w(2g-c) = g-c+\psi_w(c) = g-c$; hence we have $\mathcal{F}N_{2g-c} = N_{g-c}$. Since $\psi_w(g-c) = 0$, we have $\mathcal{V}^{-1}N_{g-c} = N_{g+(g-c)-\psi_w(g-c)} = N_{2g-c}$. □

Proposition 3.1.5. *Assume $c \leq \lfloor g/2 \rfloor$. Let M be a principally quasi-polarized Dieudonné module over k and set $w = \mathcal{E}(N) \in {}^I W_g$. The following conditions are equivalent:*

1. *$w \in {}^I W_g^{(c)}$,*
2. *M is supersingular and $S^0(M)/S_0(M)$ is a k-vector space of dimension $2c$. (In this case we have $S_0(M) = \langle\langle N_{2g-c} \rangle\rangle$.)*

Proof. Let us introduce two conditions (1′) $w \in {}^I W_g^{[c]}$ and (2′) M is supersingular and $S^0(M)/S_0(M)$ is a k-vector space of dimension $\leq 2c$. It suffices to show (1) ⇒ (2′) and (2) ⇒ (1′).

(1) ⇒ (2′): Note $\langle\langle N_{2g-c} \rangle\rangle$ is superspecial by Lemma 3.1.4; hence we have $\langle\langle N_{2g-c} \rangle\rangle \subset S_0(M)$. Lemma 3.1.4 (1) implies $\mathcal{V}M \subset \langle\langle N_{2g-c} \rangle\rangle$ and therefore $\mathcal{V}M \subset S_0(M)$ holds. Since $\mathcal{V}S^0(M)$ is the smallest superspecial Dieudonné module containing $\mathcal{V}M$, we have $\mathcal{V}S^0(M) \subset S_0(M)$. Thus $S^0(M)/S_0(M)$ is a k-vector space. Since

$$\dim M/S_0(M) \leq \dim N/N_{2g-c} \leq c,$$

we obtain $\dim S^0(M)/S_0(M) \leq 2c$.

(2) ⇒ (1′): Since $S^0(M)/S_0(M)$ is a k-vector space, we have

$$\mathcal{V}M \subset \mathcal{V}S^0(M) \subset S_0(M) \subset M.$$

Hence $\overline{S_0(M)} = (\mathcal{V}^{-1}\mathcal{F})^\infty N$ by Lemma 3.1.3. Note $\dim \overline{S_0(M)} = 2g-c$. Then the dimension of $\mathcal{F}\overline{S_0(M)}$ is greater than or equal to $g-c$. Since $\mathcal{F}^2\overline{S_0(M)} = p\overline{S_0(M)} = 0$, we have $\psi_w(g-c) = 0$. This is equivalent to $w \in {}^I W_g^{[c]}$ by Lemma 2.5.4. □

Assume $c \leq \lfloor g/2 \rfloor$. Let $w \in {}^I W_g^{(c)}$ and let v be the element of W_c determined by $v(i) = w(g-c+i) - (g-c)$ for all $1 \leq i \leq c$. Let M be any principally quasi-polarized Dieudonné module with $\mathcal{E}(M/pM) = w$. Consider the subspace $L := \mathcal{V}S^0(M)/\mathcal{V}S_0(M)$ of $N/\overline{\mathcal{V}S_0(M)}$. We put

$$b'_i = b_{g-c+i} \mod \overline{\mathcal{V}S_0(M)} \tag{16}$$

for $1 \leq i \leq 2c$. Then L is a k-vector space of dimension $2c$ with a basis $\{b'_1, \dots, b'_{2c}\}$; moreover the quasi-polarization on $S^0(M)$ induces a perfect alternating pairing on L, which satisfies

$$\langle b'_i, b'_{2c+1-j} \rangle = \begin{cases} 1 & \text{if} \quad i = j \quad \text{and} \quad v(i) > c, \\ -1 & \text{if} \quad i = j \quad \text{and} \quad v(i) \leq c, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{17}$$

Consider the two maximal totally isotropic subspaces $C = \mathcal{V}M/\mathcal{V}S_0(M)$ and $D = \mathcal{F}M/\mathcal{V}S_0(M)$ of L. Let $\mathrm{P}(M)$ (resp. $\mathrm{Q}(M)$) be the parabolic subgroup of $\mathrm{Sp}(L)$ stabilizing C (resp. D).

Proposition 3.1.6. *Assume $c \leq \lfloor g/2 \rfloor$. Let $w' \in \overline{W}'_c$. For a principally quasi-polarized Dieudonné module M with $\mathcal{E}(M/pM) \in {}^{I}W_g^{(c)}$, the following are equivalent:*

(1) *we have $\mathfrak{r}(\mathcal{E}(M/pM)) = w'$, see* (4) *for the definition of* $\mathfrak{r}$,

(2) *there exists an isomorphism $u : \mathrm{Sp}(L) \simeq \mathrm{Sp}(L_0 \otimes k)$ such that $u(\mathrm{P}(M)) = \mathrm{P}_0 \otimes k$ and $u(\mathrm{Q}(M)) = {}^{v'}(\mathrm{P}_0 \otimes k)$ for a lift $v' \in W_c$ of w'.*

Proof. Suppose (1). Put $w = \mathcal{E}(M/pM)$. Let v be as above, i.e., the element of W_c determined by $v(i) = w(g - c + i) - (g - c)$ for all $1 \leq i \leq c$. Note v is a lift of w' by the definition of $\mathfrak{r}$. It follows from Lemma 2.5.5 and (16) that C has a basis $\{b'_{v^{-1}(1)}, \dots, b'_{v^{-1}(c)}\}$ and D has a basis $\{b'_1, \dots, b'_c\}$. Hence there exists an isomorphism $u : \mathrm{Sp}(L) \simeq \mathrm{Sp}(L_0 \otimes k)$ such that $u(\mathrm{P}(M)) = \mathrm{P}_0 \otimes k$ and $u(\mathrm{Q}(M)) = {}^{v}(\mathrm{P}_0 \otimes k)$.

Conversely suppose (2). Put $w'_0 := \mathfrak{r}(\mathcal{E}(M/pM))$. By (1) $\Rightarrow$ (2), there exists an isomorphism $u_0 : \mathrm{Sp}(L) \simeq \mathrm{Sp}(L_0 \otimes k)$ such that $u_0(\mathrm{P}(M)) = \mathrm{P}_0 \otimes k$ and $u_0(\mathrm{Q}(M)) = {}^{v'_0}(\mathrm{P}_0 \otimes k)$ for a lift $v'_0 \in W_c$ of w'_0. Since any automorphism of $\mathrm{Sp}(L_0 \otimes k)$ is an inner automorphism, there exists $h \in \mathrm{Sp}(L_0)(k)$ such that $\mathrm{Ad}\, h = u_0 \circ u^{-1}$. Then ${}^{h}(\mathrm{P}_0 \otimes k) = \mathrm{P}_0 \otimes k$ and ${}^{hv'}(\mathrm{P}_0 \otimes k) = {}^{v'_0}(\mathrm{P}_0 \otimes k)$. Hence we have $v' \in \mathrm{P}_0(k) v'_0 \mathrm{P}_0(k)$, which means $w' = w'_0$. □

3.2 Polarizations

Let E be a supersingular elliptic curve over $\mathbb{F}_p$ whose $\overline{\mathbb{F}_p}$-endomorphisms are all defined over $\mathbb{F}_{p^2}$, see [26], (4.1) for the existence of such an E. We denote by F the Frobenius endomorphism of E. Let $\mathcal{O} = \mathrm{End}(E \otimes \mathbb{F}_{p^2})$ and let B denote $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$, which is the quaternion algebra over $\mathbb{Q}$ ramified only at p and ∞ over $\mathbb{Q}$. Note $\mathcal{O}$ is a maximal order of B. Let $x \mapsto \overline{x}$ denote the main involution of B.

We claim that every polarization on E^g over $\overline{\mathbb{F}_p}$ is defined over $\mathbb{F}_{p^2}$. By the definition of polarization (cf. [22], Definition 6.3), we need only prove that all $\overline{\mathbb{F}_p}$-homomorphisms from E^g to $(E^g)^t$ are defined over $\mathbb{F}_{p^2}$. Note that the divisor $E^{g-1} \times \{0\} + E^{g-2} \times \{0\} \times E + \cdots + \{0\} \times E^{g-1}$ on E^g defines a principal polarization $\eta : E^g \simeq (E^g)^t$ on E^g, which is defined over $\mathbb{F}_p$. Hence it suffices to show that all $\overline{\mathbb{F}_p}$-endomorphisms of E^g are defined over $\mathbb{F}_{p^2}$. This follows since all $\overline{\mathbb{F}_p}$-endomorphisms of E are defined over $\mathbb{F}_{p^2}$.

Let n be a natural number with $\gcd(n,p)=1$, and let c be a non-negative integer with $c \le \lfloor g/2 \rfloor$. Let $\mathcal{P}_{c,n}$ denote the set of pairs (μ,θ) of polarizations μ and level n-structures θ on E^g such that $\operatorname{Ker}\mu \simeq \alpha_p^{\oplus 2c}$. Here α_p is the finite group scheme $\operatorname{Ker}(F:\mathbb{G}_a \to \mathbb{G}_a)$. Let $\mathbb{F}$ be the finite field $\mathbb{F}_{p^2}(E[n])$. Then every element (μ,θ) of $\mathcal{P}_{c,n}$ is defined over $\mathbb{F}$. From now on we write E instead of $E \otimes \mathbb{F}$. For two elements (μ,θ) and (μ',θ') of $\mathcal{P}_{c,n}$, we say $(\mu,\theta) \approx (\mu',\theta')$ if there exists an automorphism h of E^g such that $\mu' = h^*\mu$ and $h \circ \theta' = \theta$. We write $\Lambda_{c,n} = \mathcal{P}_{c,n}/\approx$. Note $\Lambda_{c,n}$ is a finite set.

Choose an element (μ_0,θ_0) of $\Lambda_{c,n}$. Set $\varphi = \eta^{-1}\circ\mu_0$, which is an element of $\mathrm{M}_g(\mathcal{O})$. We define a quaternion unitary group G over $\mathbb{Z}$ by

$$\mathrm{G}(R) = \{h \in \mathrm{GL}_g(\mathcal{O}\otimes_{\mathbb{Z}} R) \mid {}^t\overline{h}\varphi h = \varphi\} \tag{18}$$

for any commutative unitary ring R. It follows from [18], 8.3 and 8.4 that for a prime number $l(\neq p)$, there exists a $u_l \in \mathrm{GL}_g(\mathcal{O}_l)$ such that ${}^t\overline{u_l}\varphi_l u_l = 1_g$ and there exists a $u_p \in \mathrm{GL}_g(\mathcal{O}_p)$ such that

$${}^t\overline{u_p}\varphi_p u_p = \operatorname{diag}(\underbrace{1,\ldots,1}_{g-2c},\underbrace{\begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix},\ldots,\begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}}_{c}). \tag{19}$$

Let K denote the open compact subgroup $\prod_l \mathrm{G}(\mathbb{Z}_l)$ of $\mathrm{G}(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty$ denotes the finite adele ring. (We remark that $\mathrm{G}(\mathbb{Z}_l)$ is isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Z}_l)$ for $l \neq p$.) Also K_n is defined to be the kernel of the natural homomorphism from K to $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$. It is known (cf. [13], §2 and [18], Ch. 8 and 9.12) that there is a bijection

$$\alpha: \quad \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}_n \longrightarrow \Lambda_{c,n}, \tag{20}$$

which is defined as follows: for any element γ of $\mathrm{G}(\mathbb{A}^\infty)$ there exists an $f \in \mathrm{GL}_g(B)$ with $(\deg f, n) = 1$ such that $f_l = \gamma_l\delta_l$ for some $\delta_l \in \operatorname{Ker}(\mathrm{GL}_g(\mathcal{O}_l) \to \mathrm{GL}_g(\mathcal{O}_l/n))$; then $\alpha(\gamma)$ is defined to be $(f^*\mu_0, (f|_{E^g[n]})^{-1}\circ\theta_0)$, where we see f as an element of $\operatorname{End}_{\mathbb{Q}}(E^g)^\times$ with $\operatorname{End}_{\mathbb{Q}}(E^g) := \operatorname{End}(E^g)\otimes_{\mathbb{Z}}\mathbb{Q}$; moreover if $\alpha(\gamma) = (\mu,\theta)$ then we have an isomorphism

$$\operatorname{Aut}(E^g,\mu,\theta) \longrightarrow \mathrm{G}(\mathbb{Q}) \cap \gamma\,\mathrm{K}_n\,\gamma^{-1}, \tag{21}$$

which is defined by sending h to fhf^{-1}.

3.3 The moduli space $\mathcal{T}_{\mu,\theta}(w')$

Assume $g \ge 2$. We retain the notation of §3.2. We continue to assume $c \le \lfloor g/2 \rfloor$ and $\gcd(n,p)=1$ and to write E instead of $E\otimes\mathbb{F}$. Let (μ,θ) be an element of $\Lambda_{c,n}$. Set $G = \operatorname{Ker}\mu$, which is an α-group over $\mathbb{F}$ of rank p^{2c}. We identify $(L_0, \langle\ ,\ \rangle)\otimes\mathbb{F}$ with the symplectic vector space over $\mathbb{F}$ associated to G.

Consider the moduli functor $\mathfrak{T}_{\mu,\theta}$ from the category of $\mathbb{F}$-schemes to the category of sets, sending S to the set of isogenies

$$\rho : (E^g, \mu, \theta) \times_{\mathbb{F}} S \to (Y, \lambda, \vartheta) \tag{22}$$

as polarized abelian schemes (i.e., $\mu_S = \rho^* \lambda$ and $\rho \circ \theta_S = \vartheta$) such that λ is a principal polarization.

Lemma 3.3.1. *$\mathfrak{T}_{\mu,\theta}$ is represented by an $\mathbb{F}$-scheme $\mathcal{T}_{\mu,\theta}$, and there is an $\mathbb{F}$-isomorphism from $\mathcal{T}_{\mu,\theta}$ to the flag variety $\mathrm{X} = \mathrm{X}_{\mathbb{F}}$.*

Proof. To give an element $\rho \in \mathfrak{T}_{\mu,\theta}(S)$ is equivalent to giving an α-subgroup H of G_S with $\mathrm{rk}\, H = p^c$ such that μ descends to a polarization on E^g_S/H. Associated to $H \subset G_S$ of rank p^c we have a locally free subsheaf $\mathcal{I}$ of rank c of $\pi^* L_0$, where $\pi : S \to \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(\mathbb{F}_{p^2})$. By Lemma 2.3.1, μ descends to a polarization on E^g_S/H if and only if $\mathcal{I}$ is totally isotropic in $\pi^* L_0$. Thus we obtain an isomorphism $\mathfrak{T}_{\mu,\theta}(S) \simeq \mathrm{X}(S)$, which is functorial on S. Thus $\mathfrak{T}_{\mu,\theta}$ is represented by X. □

There is a canonical morphism Ψ from $\mathcal{T}_{\mu,\theta}$ to the supersingular locus $\mathcal{W}_{\sigma,n}$ defined by sending $\rho : (E^g, \mu, \theta) \to (Y, \lambda, \vartheta)$ to (Y, λ, ϑ). Let w' be an element of $\overline{W}'_c$. Let $\mathcal{T}_{\mu,\theta}(w')$ be the subset of $\mathcal{T}_{\mu,\theta}$ consisting of $\rho : (E^g, \mu, \theta) \to (Y, \lambda, \vartheta)$ with $\mathcal{E}(Y[p]) \in {}^I W_g^{(c)}$ and $\mathfrak{r}(\mathcal{E}(Y[p])) = w'$. By [27], (3.2), $\mathcal{T}_{\mu,\theta}(w')$ is locally closed in $\mathcal{T}_{\mu,\theta}$; we consider this as a locally closed subscheme in $\mathcal{T}_{\mu,\theta}$ by giving it the reduced induced scheme structure. Let $\mathcal{J}_{w',n}$ be the subset of $\mathcal{A}_{g,n}$ defined by

$$\mathcal{J}_{w',n} = \bigcup_{\mathfrak{r}(w)=w'} \mathcal{S}_{w,n}.$$

Note $\mathcal{J}_{w',n}$ is a locally closed subset of $\mathcal{A}_g$; we give it the reduced induced scheme structure.

Proposition 3.3.2. *Let $w' \in \overline{W}'_c$ with $c \le \lfloor g/2 \rfloor$.*

(1) *For every $(\mu, \theta) \in \Lambda_{c,n}$ there is an isomorphism over $\mathbb{F}$:*

$$\mathcal{T}_{\mu,\theta}(w') \longrightarrow \mathrm{X}(w').$$

(2) *Ψ induces a finite surjective morphism over $\mathbb{F}$:*

$$\overline{\Psi} : \quad \coprod_{(\mu,\theta)\in\Lambda_{c,n}} \mathrm{Aut}(E^g, \mu, \theta)\backslash \mathcal{T}_{\mu,\theta}(w') \longrightarrow \mathcal{J}_{w',n},$$

which is bijective on geometric points. If $n \ge 3$, then Ψ induces an isomorphism

$$\tilde{\Psi} : \quad \coprod_{(\mu,\theta)\in\Lambda_{c,n}} \mathcal{T}_{\mu,\theta}(w') \longrightarrow \tilde{\mathcal{J}}_{w',n},$$

where $\tilde{\mathcal{J}}_{w',n}$ is the normalization of $\mathcal{J}_{w',n}$.

Proof. (1) Let k be an algebraically closed field of characteristic p. Let $\rho : (E^g, \mu, \theta) \otimes_{\mathbb{F}} k \to (Y, \lambda, \vartheta)$ be an element of $\mathcal{T}_{\mu,\theta}(k)$. The element ρ defines an isogeny $M \subset M_{1,k}$ with $M_{1,k} := M_1 \otimes W(k)$, where $M = \mathbb{D}(Y)$ and $M_1 = \mathbb{D}(E^g)$. It follows from Proposition 3.1.5 that if $\rho \in \mathcal{T}_{\mu,\theta}(w')(k)$, then $M \subset M_{1,k}$ is a minimal isogeny. Let $\mathrm{P}(M)$ and $\mathrm{Q}(M)$ be the parabolic subgroups of $\mathrm{Sp}(\mathcal{V}M_1/\mathcal{V}M_1^t) \otimes k$ stabilizing the maximal totally isotropic subspaces $\mathcal{V}M/\mathcal{V}M_{1,k}^t$ and $\mathcal{F}M/\mathcal{V}M_{1,k}^t$ respectively. Recall that we identify $L_0 \otimes \mathbb{F}$ with the locally free sheaf over $\mathrm{Spec}(\mathbb{F})$ associated to $G = \mathrm{Ker}\,\mu$; then we have a canonical isomorphism $\nu : \mathrm{Sp}(L_0 \otimes \mathbb{F}) \simeq \mathrm{Sp}(\mathcal{V}M_1/\mathcal{V}M_1^t)$. Now we regard $\mathrm{P}(M)$ and $\mathrm{Q}(M)$ as subgroups of $\mathrm{Sp}(L_0 \otimes k)$ via ν. Then the isomorphism $\mathcal{T}_{\mu,\theta} \simeq \mathrm{X}$ obtained in Lemma 3.3.1 sends ρ to $\mathrm{P}(M)$. By Proposition 3.1.6 we have $\rho \in \mathcal{T}_{\mu,\theta}(w')(k)$ if and only if there exists an $h \in \mathrm{Sp}(L_0)(k)$ such that ${}^h\,\mathrm{P}(M) = \mathrm{P}_0 \otimes k$ and ${}^h\,\mathrm{Q}(M) = {}^{v'}(\mathrm{P}_0 \otimes k)$ for a lift $v' \in W_c$ of w'. Here we used the fact that every automorphism of $\mathrm{Sp}(L_0) \otimes k$ is an inner automorphism. We also have $\mathrm{Fr}\,\mathrm{P}(M) = \mathrm{Q}(M)$. Hence, identifying $\mathcal{T}_{\mu,\theta}$ with X, we obtain $\mathcal{T}_{\mu,\theta}(w')(k) = \mathrm{X}(w')(k)$ in $\mathrm{X}(k)$. Since both $\mathcal{T}_{\mu,\theta}(w')$ and $\mathrm{X}(w')$ are reduced locally closed subschemes in X, we have $\mathcal{T}_{\mu,\theta}(w') = \mathrm{X}(w')$ by the Hilbert Nullstellensatz.

(2) Let T_n denote the source of $\overline{\Psi}$. First we show that $\overline{\Psi}$ sets up a bijection from $T_n(k)$ to $\mathcal{J}_{w',n}(k)$ for any algebraically closed field k. By Proposition 3.1.5, for any $(Y, \lambda, \vartheta) \in \mathcal{J}_{w',n}(k)$ we have an isogeny $S_0(M) \subset M$ with $M = \mathbb{D}(Y)$ such that $M/S_0(M)$ is a k-vector space of dimension c; by [24], Theorem 6.2 we have a corresponding isogeny $\rho : E^g \to Y$ with $\mathrm{Ker}\,\rho \simeq \alpha_p^c$; then setting $\mu = \rho^*\lambda$ and $\theta = (\rho|_{E^g[n]})^{-1} \circ \vartheta$, we have a geometric point $\rho : (E^g, \mu, \theta) \to (Y, \lambda, \vartheta)$ of $\mathcal{T}_{\mu,\theta}(w')$. Proposition 3.1.5 also says that every element ρ of $\mathcal{T}_{\mu,\theta}(w')(k)$ is a minimal isogeny. Hence $\overline{\Psi}$ is bijective on geometric points by Lemma 2.4.2.

By definition $\mathcal{T}_{\mu,\theta}(w')$ is the reduced scheme associated to $\mathcal{T}_{\mu,\theta} \times_{\mathcal{W}_{\sigma,n}} \mathcal{J}_{w',n}$. Since $\Psi : \mathcal{T}_{\mu,\theta} \to \mathcal{W}_{\sigma,n}$ is proper, the composition

$$\Psi' : \mathcal{T}_{\mu,\theta}(w') \hookrightarrow \mathcal{T}_{\mu,\theta} \times_{\mathcal{W}_{\sigma,n}} \mathcal{J}_{w',n} \to \mathcal{J}_{w',n}$$

is proper. Clearly Ψ' is quasi-finite; hence this is finite. Since $\mathcal{J}_{w',n}$ is noetherian, the induced morphism $\mathrm{Aut}(E^g, \mu, \theta)\backslash\mathcal{T}_{\mu,\theta}(w') \to \mathcal{J}_{w',n}$ is finite. Thus $\overline{\Psi}$ is finite.

Assume $n \geq 3$. Then we have $\mathrm{Aut}(E^g, \mu, \theta) = \{\mathrm{id}\}$. Since $\mathcal{T}_{\mu,\theta}(w')$ is regular by (1), $\overline{\Psi}$ induces a morphism

$$\tilde{\Psi} : \coprod_{(\mu,\theta)\in\Lambda_{c,n}} \mathcal{T}_{\mu,\theta}(w') \longrightarrow \tilde{\mathcal{J}}_{w',n}.$$

By [10], §8, Lemme (8.12.10.1), it suffices to check that $\tilde{\Psi}$ is birational, in order to show that $\tilde{\Psi}$ is an isomorphism. Let (Y, λ, ϑ) be the polarized abelian variety with level n-structure over a generic point $\mathrm{Spec}(K) \to \mathcal{J}_{w',n}$. By Lemma 2.4.2 there is an isogeny $\rho : (Z, \mu', \theta') \to (Y, \lambda, \vartheta)$ such that ρ is a minimal isogeny. Since $\mathcal{A}_{g,p^c,n}$ is a fine moduli space, (Z, μ', θ') can be written as $(E^g, \mu, \theta) \otimes_{\mathbb{F}} K$ for some $(\mu, \theta) \in \Lambda_{c,n}$. Hence by associating ρ to (Y, λ, ϑ), we obtain the inverse morphism of $\tilde{\Psi}$ on generic points. Thus $\tilde{\Psi}$ is birational. □

3.4 The main theorem

Assume $c \le \lfloor g/2 \rfloor$. Let n be a natural number with $\gcd(n,p)=1$. Let (μ_0,θ_0) be the element of $\Lambda_{c,n}$ chosen in §3.2. Let $w' \in \overline{W}'_c$. We identify $\mathrm{X}(w')$ with $\mathcal{T}_{\mu_0,\theta_0}(w')$ and define the action on $\mathrm{X}(w')$ of $\mathrm{G}(\mathbb{Q})$ by the natural action on $\mathcal{T}_{\mu_0,\theta_0}(w')$ of $\mathrm{G}(\mathbb{Q}) = \{h \in \mathrm{End}_{\mathbb{Q}}(E^g)^\times \mid h^*(\mu_0)=\mu_0\}$.

Theorem 3.4.1. *There is a finite surjective morphism*

$$\Phi: \quad \mathrm{G}(\mathbb{Q})\backslash(\mathrm{X}(w')\times \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}_n) \longrightarrow \mathcal{J}_{w',n}$$

over $\mathbb{F}=\mathbb{F}_{p^2}(E[n])$, which is bijective on geometric points. If $n \ge 3$, then Φ induces an isomorphism $\tilde{\Phi}: \mathrm{G}(\mathbb{Q})\backslash(\mathrm{X}(w')\times\mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}_n) \to \tilde{\mathcal{J}}_{w',n}$.

Proof. Clearly the left hand side can be written as

$$\coprod_{\gamma\in \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}_n} \Gamma_\gamma\backslash \mathrm{X}(w') \tag{23}$$

with $\Gamma_\gamma = \mathrm{G}(\mathbb{Q})\cap\gamma\,\mathrm{K}_n\,\gamma^{-1}$. If we write $(\mu,\theta)=\alpha(\gamma)$, then Γ_γ is identified with $\mathrm{Aut}(E^g,\mu,\theta)$. Hence the theorem is nothing but Proposition 3.3.2. □

3.5 Reducibility of supersingular Ekedahl-Oort strata

Assume $c \le \lfloor g/2\rfloor$. Let $W_{c,J}$ be the subgroup of W_c generated by the elements of $J=\{s_1,\dots,s_{c-1}\}$. Let w' be an element of $\overline{W}'_c$. Note a (any) representative of w' is not in $W_{c,J}$. Hence by Bonnafé and Rouquier [1], Theorem 2, the Deligne-Lusztig variety $\mathrm{X}(w')$ is irreducible, since $W_{c,J}$ is a maximal parabolic subgroup of W_c. Thus from Theorem 3.4.1 and (23), we have

Corollary 3.5.1. *The set of irreducible (connected) components of $\mathcal{J}_{w'}$ is identified with $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}$.*

Let us estimate $H_{g,c} = \sharp\,\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}$ by the mass formula. We have

$$H_{g,c} \ge 2\mathfrak{m}_{g,c},$$

where

$$\mathfrak{m}_{g,c} = \sum_{\gamma\in\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}} \frac{1}{\sharp\,\mathrm{G}(\mathbb{Q})\cap\gamma\,\mathrm{K}\,\gamma^{-1}}.$$

From now on, we compute the mass $\mathfrak{m}_{g,c}$. Applying Prasad's mass formula [28] to G, we have

$$\mathfrak{m}_{g,c} = \prod_{i=1}^{g}\frac{(2i-1)!}{(2\pi)^{2i}}\cdot\prod_{l\neq p}\frac{l^{2g^2+g}}{\sharp\,\mathrm{Sp}_{2g}(\mathbb{F}_l)}\cdot\frac{p^{(\dim \mathrm{L}_p+2g^2+g)/2}}{\sharp\,\mathrm{L}_p(\mathbb{F}_p)}, \tag{24}$$

where L_p is a connected subgroup scheme over $\mathbb{F}_p$ of $\mathrm{G}_p = \mathrm{G}\otimes_{\mathbb{Z}}\mathbb{F}_p$ such that

$$\mathrm{G}_p = \mathrm{L}_p\cdot\mathcal{R}_u(\mathrm{G}_p) \quad\text{with}\quad \mathrm{L}_p\cap\mathcal{R}_u(\mathrm{G}_p)=\{1\} \tag{25}$$

with the unipotent radical $\mathcal{R}_u(\mathrm{G}_p)$ of G_p (cf. [2, 11.21], $\mathcal{R}_u(\mathrm{G}_p)$ is a reduced subgroup scheme over $\mathbb{F}_p$ of G_p and for an algebraically closed field k, $\mathcal{R}_u(\mathrm{G}_p)\otimes k$ is the unipotent part of the connected component of the intersection $\bigcap \mathrm{B}$ of all Borel k-subgroups B of $\mathrm{G}_p \otimes k$). We call (25) a *Levi decomposition* of G_p. Note (24) is independent of the choice of Levi decomposition.

We need to choose a Levi decomposition and describe it explicitly. Put $\mathfrak{o} = \mathcal{O} \otimes \mathbb{F}_p$, which can be written as $\mathbb{F}_{p^2}[F]/(F^2 = 0, Fa = {}^{\sigma}aF,\ a \in \mathbb{F}_{p^2})$. The main involution of $\mathcal{O}$ induces an involution of $\mathfrak{o}$, which sends $x = a + bF$ to $\overline{x} = {}^{\sigma}a - bF$ for $a, b \in \mathbb{F}_{p^2}$. By (19) there is an isomorphism from G_p to the affine group scheme $\mathcal{G}_p$ defined by

$$\mathcal{G}_p(R) = \{h \in \mathrm{GL}_g(\mathfrak{o} \otimes R) \mid {}^t\overline{h}\psi h = \psi\}$$

for any $\mathbb{F}_p$-algebra R, where ψ is the right hand side of (19) regarded as an element of $\mathrm{M}_g(\mathfrak{o})$. We can define a subgroup scheme $\mathcal{N}_p$ over $\mathbb{F}_p$ of $\mathcal{G}_p$ by the functor

$$R \quad \longmapsto \quad \mathcal{G}_p(R) \cap (1 + F\,\mathrm{M}_g(\mathbb{F}_{p^2} \otimes R)) \tag{26}$$

for any $\mathbb{F}_p$-algebra R; indeed let $u \in \mathrm{M}_g(\mathbb{F}_{p^2} \otimes R)$ and write $u = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{M}_{g-2c}(\mathbb{F}_{p^2} \otimes R)$; then $1 + Fu$ is in $\mathcal{N}_p(R)$ if and only if $A = {}^tA$ and $B = 0$; hence the functor (26) is represented by an $\mathbb{F}_p$-scheme (a subgroup scheme of $\mathcal{G}_p$). Note that $\mathcal{N}_p$ is geometrically connected, and this is a unipotent normal subgroup of $\mathcal{G}_p$. We can also define a subgroup scheme $\mathcal{L}_p$ over $\mathbb{F}_p$ of $\mathcal{G}_p$ by the functor

$$R \quad \longmapsto \quad \{h \in \mathrm{GL}_g(\mathbb{F}_{p^2} \otimes R) \mid h^{\dagger}\psi h = \psi \text{ in } \mathrm{M}_g(\mathfrak{o} \otimes R)\} \tag{27}$$

for any $\mathbb{F}_p$-algebra R, where $h^{\dagger} := {}^t({}^{\sigma}h)\ (= {}^t\overline{h})$; indeed we have

Lemma 3.5.2. *The functor* (27) *is represented by* $\mathrm{U}_{g-2c} \times \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathrm{Sp}_{2c}$*, where* U_m *is the unitary group and* $\mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathrm{Sp}_{2m}$ *is the Weil restriction of the symplectic group:*

$$\begin{aligned} \mathrm{U}_m(R) &= \{A \in \mathrm{GL}_m(\mathbb{F}_{p^2} \otimes R) \mid A^{\dagger}A = 1_m\}, \\ \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathrm{Sp}_{2m}(R) &= \{D \in \mathrm{GL}_{2m}(\mathbb{F}_{p^2} \otimes R) \mid {}^tDJD = J\} \end{aligned}$$

for any $\mathbb{F}_p$*-algebra* R *with*

$$J = \mathrm{diag}(\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{m}).$$

In particular $\mathcal{L}_p$ *is a connected reductive algebraic group.*

Proof. Let R be any $\mathbb{F}_p$-algebra and set $R' = \mathbb{F}_{p^2} \otimes R$. Let $h \in \mathrm{GL}_g(R')$. Write $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{M}_{g-2c}(R')$ and $D \in \mathrm{M}_{2c}(R')$. Then the condition $h^{\dagger}\psi h = \psi$ can be paraphrased as $A^{\dagger}A = 1_{g-2c}$ and ${}^tDJD = J$ with $B = 0$ and $C = 0$. □

By (26) and (27) we have

$$\mathcal{G}_p = \mathcal{L}_p \cdot \mathcal{N}_p \quad \text{with} \quad \mathcal{L}_p \cap \mathcal{N}_p = \{1\}. \tag{28}$$

Lemma 3.5.3. (28) *is a Levi decomposition of* $\mathcal{G}_p$.

Proof. It suffices to show that $\mathcal{N}_p$ is the unipotent radical of $\mathcal{G}_p$. Let k be an algebraically closed field. We first prove that $\mathcal{N}_p \otimes k$ is contained in every Borel subgroup of $\mathcal{G}_p \otimes k$. Let B_0 be a Borel subgroup containing $\mathcal{N}_p \otimes k$. Let B be any Borel subgroup of $\mathcal{G}_p \otimes k$. By [2, 11.1], B is conjugate to B_0, say $\mathrm{B} = {}^h\,\mathrm{B}_0$ for a certain $h \in \mathcal{G}_p(k)$. Hence $\mathcal{N}_p \otimes k = {}^h(\mathcal{N}_p \otimes k) \subset {}^h\,\mathrm{B}_0 = \mathrm{B}$. Since $\mathcal{N}_p \otimes k$ is connected and unipotent, we have $\mathcal{N}_p \subset \mathcal{R}_u(\mathcal{G}_p)$. Applying [2, 14.11] to the homomorphism $f : \mathcal{G}_p \to \mathcal{G}_p/\mathcal{N}_p \simeq \mathcal{L}_p$, we have $f(\mathcal{R}_u(\mathcal{G}_p)) = \mathcal{R}_u(\mathcal{L}_p)$. Since $\mathcal{L}_p$ is reductive (Lemma 3.5.2), we have $\mathcal{R}_u(\mathcal{L}_p) = \{1\}$. Hence we obtain $\mathcal{R}_u(\mathcal{G}_p) \subset \mathcal{N}_p$. □

Proposition 3.5.4. *We have*

$$\mathfrak{m}_{g,c} = \prod_{i=1}^{g} \frac{(2i-1)!\zeta(2i)}{(2\pi)^{2i}} \cdot \binom{g}{2c}_{p^2} \cdot \prod_{i=1}^{g-2c} (p^i + (-1)^i) \prod_{i=1}^{c} (p^{4i-2} - 1),$$

where $\zeta(s)$ *is the Riemann zeta function and*

$$\binom{g}{r}_q := \frac{\prod_{i=1}^{g}(q^i - 1)}{\prod_{i=1}^{r}(q^i - 1) \prod_{i=1}^{g-r}(q^i - 1)} \quad \in \mathbb{Z}[q].$$

Proof. As L_p in (24) we can take the group isomorphic to $\mathcal{L}_p$ via the isomorphism $\mathrm{G}_p \simeq \mathcal{G}_p$. Then we have $\dim \mathrm{L}_p = (g-2c)^2 + 2(2c^2 + c)$. The desired equation follows from (24) and the formulas

$$\sharp\,\mathrm{Sp}_{2m}(\mathbb{F}_q) = q^{2m^2+m} \prod_{i=1}^{m} (1 - q^{-2i}) \quad \text{and} \quad \sharp\,\mathrm{U}_m(\mathbb{F}_q) = q^{m^2} \prod_{i=1}^{m} (1 - (-1)^i q^{-i})$$

(cf. [4, Chapter 1], where the notation $\mathrm{U}_m(\mathbb{F}_{q^2})$ is used instead of $\mathrm{U}_m(\mathbb{F}_q)$). □

Corollary 3.5.5. *If* $w \in {}^I W_g^{(c)}$ *with* $c \le \lfloor g/2 \rfloor$, *then* $\mathcal{S}_w$ *is reducible except possibly for small* p.

Proof. Set $w' = \mathfrak{r}(w)$. Corollary 3.5.1 says that the Hecke action on the set of connected components of $\mathcal{J}_{w'}$ is transitive. Since the Hecke action stabilizes $\mathcal{S}_w$, the number of connected components of $\mathcal{S}_w$ is greater than or equal to that of $\mathcal{J}_{w'}$. Clearly we have $\lim_{p\to\infty} H_{g,c} \ge \lim_{p\to\infty} \mathfrak{m}_{g,c} = \infty$. Hence $\mathcal{S}_w$ is reducible for sufficiently large p. □

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