On the affineness of distinguished Deligne-Lusztig varieties

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Abstract

We consider Deligne-Lusztig varieties in the variety of parabolic subgroups with a fixed type. In this paper, we give a criterion for the affineness of distinguished Deligne-Lusztig varieties, extending and refining the original Deligne-Lusztig criterion. In particular we show that distinguished Deligne-Lusztig varieties are affine except possibly for small q, and that all distinguished Deligne-Lusztig varieties associated to rank-2 groups are affine.

1 Introduction

Let p be a prime. Let k_0 be a finite field of characteristic p. Let k be an algebraic closure of k_0. Let G_0 be a connected reductive algebraic group over k_0 and set $G = G_0 \times \mathrm{Spec}(k)$. Let $frob$ be the Frobenius map on G_0. Let F be an endomorphism of G over k such that $F^d = frob \times \mathrm{Spec}(k)$ for some d. Let q be the positive real number with $q^d = |k_0|$.

We fix an F-stable Borel subgroup B and a maximal torus T contained in B. Let W be the Weyl group $N_G(T)/T$. We write $\dot{w}$ for a representative in $N_G(T)$ of $w \in W$. Let Φ denote the set of roots. Let Φ^+ (resp. Φ^-) be the set of positive roots (resp. the set of negative roots) with respect to B. We denote by Δ the set of simple roots. Let U_α be the root group associated to $\alpha \in \Phi$. The endomorphism F induces a permutation σ of Φ so that F sends U_α to $U_{\sigma(\alpha)}$. Since B is F-stable, σ stabilizes Φ^+ and hence Δ.

Let I be a subset of Δ. Write W_I for the subgroup of W generated by the simple reflections s_α associated to $\alpha \in I$. We denote by P_I the standard parabolic subgroup BW_IB. A parabolic subgroup of G is called of type I if it is conjugate to P_I. Let X_I be the set of parabolic subgroups of type I, which has a canonical structure of a smooth projective k-scheme. Let J be another subset of Δ. We write ${}^g x$ for gxg^{-1} for $g, x \in G$. Consider the diagonal action of G on $X_I \times X_J$ and let $O_{IJ}(w)$ denote the orbit of $(P_I, {}^{\dot{w}}P_J)$. Then we have

$$X_I \times X_J = \coprod_{w \in W_I \backslash W / W_J} O_{IJ}(w). \tag{1}$$

For $w \in W_I \backslash W / W_J$, we denote by $\tilde{w}$ the minimal-length representative in w (cf. [4], Ch. IV, Ex. §1, 3). The orbit $O_{IJ}(w)$ is called *distinguished* if $I = \tilde{w}J$. This is equivalent to that there exists a representative v in w satisfying $I = vJ$ (such v turns out to be $\tilde{w}$).

The (generalized) Deligne-Lusztig variety $X_I(w)$ associated to $w \in W_I \backslash W / W_{\sigma I}$ is the locally closed subscheme of X_I consisting of parabolic subgroups P such that $(P, F(P)) \in O_{I,\sigma I}(w)$. In other words $X_I(w)$ is the intersection of $O_{I,\sigma I}(w)$ and the graph of F. We call $X_I(w)$ *distinguished* if $I = \tilde{w}\sigma I$.

The affineness of $X_I(w)$ is one of our main concern. In the case of $I = \emptyset$, we have several criterions for the affineness. The original paper [5] has already provided a strong combinatorial criterion ([5], Theorem 9.7), which in particular implies that $X(w)$ is affine if q is at least the Coxeter number. Orlik and Rapoport [11] conjectured that $X(w)$ is affine if w is minimal in its F-conjugacy class, with a proof in the case of the split classical groups. This conjecture was proved in general by He [8]. Bonnafé and Rouquier [3] found a new criterion, which implies Orlik-Rapoport conjecture. Here we remark that no non-affine Deligne-Lusztig variety has been found so far. However for general I (without $I = \tilde{w}\sigma I$), there are many examples of non-affine $X_I(w)$ (cf. [2], Introduction). This would come from that the decomposition of the flag variety X_I into the Deligne-Lusztig varieties is coarse. In [2] Bédard pointed out this and introduced a finer decomposition:

$$X_I = \coprod_{w \in {}^I W} X_I^{\mathrm{fin}}(w) \tag{2}$$

with the set ${}^I W$ of minimal-length representatives of $W_I \backslash W$, and showed that $X_I^{\mathrm{fin}}(w)$ is isomorphic to the distinguished Deligne-Lusztig variety $X_J(w)$ with $J := \bigcap_{n \geq 0} (w\sigma)^n I$, see [2], II, 13.

Now it would be natural to expect that almost all distinguished Deligne-Lusztig varieties are affine. In this paper, we confirm it by extending the Deligne-Lusztig criterion to our general parabolic case, and also refine a little bit the criterion (even in the case $I = \emptyset$) in order to include all distinguished Deligne-Lusztig varieties in rank-2 groups.

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2 The generalized Deligne-Lusztig sections

In this section, we construct a global section of a certain invertible sheaf on $O_{IJ}(w)$ and investigate a prolongation of this section to the Zariski closure of $O_{IJ}(w)$ in $X_I \times X_J$.

2.1 Construction

Let I be a subset of Δ. We denote by L_I the standard Levi subgroup of P_I, which is the centralizer in G of T_I, where T_I is the identity component of $\bigcap_{\alpha \in I} \mathrm{Ker}(\alpha)$. For an algebraic group H, let $\mathrm{X}(H)$ denote the character group $\mathrm{Hom}(H, \mathbb{G}_m)$ and let $\mathrm{Y}(H)$ denote the cocharacter group $\mathrm{Hom}(\mathbb{G}_m, H)$. We freely use the identifications

$$\mathrm{X}(P_I) = \mathrm{X}(L_I) = \{\lambda \in \mathrm{X}(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for } \alpha \in I\}, \tag{3}$$

which are induced by the restriction maps (cf. [9], II, 1.18). Here $\langle\ ,\ \rangle$ is the canonical pairing $\mathrm{X}(T) \times \mathrm{Y}(T) \to \mathbb{Z}$ and $\alpha^\vee \in \mathrm{Y}(T)$ is the coroot of α (cf. [5], 5.1).

Let $w \in W$. Let $\lambda \in \mathrm{X}(P_I)$. Consider the morphism $\rho : G \to O_{IJ}(w)$ sending g to ${}^g(P_I, {}^w P_J)$, which is a $P_I \cap {}^w P_J$-torsor on $O_{IJ}(w)$. We define an invertible sheaf $\mathcal{E}^w_{IJ}(\lambda)$ on $O_{IJ}(w)$ by

$$\mathcal{E}^w_{IJ}(\lambda)(V) = \{f \in \mathcal{O}_{\rho^{-1}(V)}(\rho^{-1}(V)) \mid f(gx) = \lambda(x)^{-1} f(g) \text{ for all } x \in P_I \cap {}^w P_J\} \tag{4}$$

for any open subscheme V of $O_{IJ}(w)$. We have a commutative diagram:

$$\begin{array}{ccccc}
G & \xrightarrow{\ \rho\ } & O_{IJ}(w) & \xrightarrow{\ \mathrm{pr}_1\ } & X_I \\
\downarrow{\scriptstyle \cdot \dot{w}} & & \downarrow{\scriptstyle \mathrm{sw}} & \searrow^{\mathrm{pr}_2} & \\
G & \longrightarrow & O_{JI}(w^{-1}) & \longrightarrow & X_J,
\end{array}$$

where sw sends $(P,Q) \in O_{IJ}(w)$ to $(Q,P) \in O_{JI}(w^{-1})$.

Lemma 2.1.1. *There exists an isomorphism of invertible sheaves on* $O_{IJ}(w)$

$$\mathcal{E}^w_{IJ}(\lambda) \longrightarrow \mathrm{sw}^* \, \mathcal{E}^{w^{-1}}_{JI}(w^{-1}\lambda).$$

Proof. Let V be an open subscheme of $O_{IJ}(w)$. Let $f_1 \in \mathcal{E}^w_{IJ}(\lambda)(V)$. To f_1 we can associate $f_2 \in \mathcal{E}^{w^{-1}}_{JI}(w^{-1}\lambda)(\mathrm{sw}(V))$ as follows. For any $h \in G$ mapped into $\mathrm{sw}(V)$ we set $f_2(h) = f_1(h\dot{w}^{-1})$. This defines an element of $\mathrm{sw}^* \, \mathcal{E}^{w^{-1}}_{JI}(w^{-1}\lambda)$, since $G \to O_{JI}(w^{-1})$ is a $P_J \cap {}^{w^{-1}}P_I$-torsor and we have $f_2(hy) = f_1(h\dot{w}^{-1}\,\mathrm{ad}(\dot{w})(y)) = \lambda(\mathrm{ad}(\dot{w})(y))^{-1} f_1(h\dot{w}^{-1}) = (w^{-1}\lambda)(y)^{-1} f_2(h)$ for any $y \in P_J \cap {}^{w^{-1}}P_I$. □

Consider the L_I-torsor

$$\pi : G/U_I \longrightarrow X_I. \tag{5}$$

Let $\mathcal{L}_I(\lambda)$ be the invertible sheaf on X_I defined by

$$\mathcal{L}_I(\lambda)(V) = \{ f \in \mathcal{O}_{\pi^{-1}(V)}(\pi^{-1}(V)) \mid f(gx) = \lambda(x)^{-1} f(g) \text{ for all } x \in L_I \} \tag{6}$$

for any open subscheme V of X_I.

Proposition 2.1.2. *Assume that P_I and ${}^wP_J$ have a common Levi subgroup. For any $\lambda \in \mathrm{X}(P_I)$, we have an isomorphism $\Psi(\dot{w}) : \mathrm{pr}_1^* \, \mathcal{L}_I(\lambda) \to \mathrm{pr}_2^* \, \mathcal{L}_J(w^{-1}\lambda)$ on $O_{IJ}(w)$.*

Proof. We claim that there exists a canonical isomorphism $\mathcal{E}^w_{IJ}(\lambda) \simeq \mathrm{pr}_1^* \, \mathcal{L}_I(\lambda)$. Indeed let V be any open subscheme of $O_{IJ}(w)$ and set $V' = \mathrm{pr}_1(V)$. The morphism $\rho^{-1}(V) \to \pi^{-1}(V') \times_{V'} V$ (an open base-change of $G \to G/U_I \times_{X_I} O_{IJ}(w)$) is the quotient by $U_I \cap {}^wP_J$. We also have $P_I \cap {}^wP_J = L_I(U_I \cap {}^wP_J)$. Hence we have

$$\begin{aligned} \mathrm{pr}_1^* \, \mathcal{L}_I(\lambda)(V) &= \{ f \in \mathcal{O}_{\pi^{-1}(V')}(\pi^{-1}(V')) \mid f(gx) = \lambda(x)^{-1} f(g), x \in L_I \} \otimes_{\mathcal{O}_{V'}(V')} \mathcal{O}_V(V) \\ &\simeq \{ h \in \mathcal{O}_{\rho^{-1}(V)}(\rho^{-1}(V)) \mid h(gx) = \lambda(x)^{-1} h(g), x \in P_I \cap {}^wP_J \} \\ &= \mathcal{E}^w_{IJ}(\lambda)(V). \end{aligned}$$

Similarly we have $\mathrm{sw}^* \, \mathcal{E}^{w^{-1}}_{JI}(w^{-1}\lambda) \simeq \mathrm{pr}_2^* \, \mathcal{L}_J(w^{-1}\lambda)$. Hence the proposition follows from Lemma 2.1.1. □

Remark 2.1.3. (1) If $I = wJ$, then P_I and ${}^wP_J$ have a common Levi subgroup, see [2], II, 7.

(2) For a $t \in T$, we have $\Psi(t\dot{w}) = \lambda(t)\Psi(\dot{w})$. We do not need to care about the choice of $\dot{w}$, since the multiplication by non-zero constant causes nothing in our arguments. We remark also that $\Psi(\dot{w})$ (up to constant multiplication) depends only on the class of w in $W_I \backslash W / W_J$.

2.2 Prolongation

Let $w \in W$. Assume $I = wJ$. We fix a reduced expression of w. Let v be any element of $W_I w W_J$. We choose a reduced expression $v = s_1 \cdots s_\ell$ such that for some c, d ($1 \le c \le d \le \ell$) we have $s_i \in W_I$ for $i < c$ and $s_i \in W_J$ for $i > d$ and $w = s_c \cdots s_d$ that is the fixed reduced expression of w, see [4], Ch. IV, Ex. §1, 3. (One may take $d = \ell$, since we have $W_I w W_J = W_I w$ by the assumption $I = wJ$.) We write $v = v_i s_i v_i'$, where $v_i = s_1 \cdots s_{i-1}$ and $v_i' = s_{i+1} \cdots s_\ell$. Let $\alpha_i \in \Delta$ be the root associated to s_i. The set $\Phi^+(v) = \Phi^+ \cap v\Phi^-$ is equal to $\{v_i\alpha_i\}_{i=1}^{\ell}$ (cf. [4], Ch. VI, §1, 6, Cor. 2). Let

$$\kappa : \quad \Phi^+(v) \longrightarrow W_I \backslash W / W_J \tag{7}$$

be the map sending $v_i\alpha_i$ to the class of v_iv_i'.

Let $\langle I\rangle$ be the submodule of $\mathrm{X}(T)$ generated by elements of I. Set $\Sigma_I = \Phi \setminus \langle I\rangle$ and $\Sigma_I^{\pm} = \Phi^{\pm} \setminus \langle I\rangle$. Put $\Sigma_I^+(v) := \Sigma_I^+ \cap v\Sigma_J^-$. Let

$$D_I^0(v) = \{\lambda \in \mathrm{X}(P_I)\otimes\mathbb{Q} \mid \langle\lambda,\alpha^\vee\rangle > 0 \text{ for } \alpha\in\Sigma_I^+(v)\} \tag{8}$$

and $D_I(v)$ the set defined by replacing $>$ in (8) by $\geq$. For $\lambda \in D_I(v)$, we set

$$R_I^\lambda(v) = \left\{\alpha\in\Sigma_I^+(v) \mid \langle\lambda,\alpha^\vee\rangle > 0\right\} \tag{9}$$

and put $\partial_I^\lambda(v) := \kappa(R_I^\lambda(v))$. As $W_I\cdot\Sigma_I^+(v)$ is independent of the choice of $v\in W_IwW_J$, so are $D_I^0(v)$ and $W_I\cdot R_I^\lambda(v)$. Also $\partial_I^\lambda(v)$ is independent of the choice of v and its reduced expression as above.

Proposition 2.2.1. *Let $\lambda\in\mathrm{X}(P_I)$. The isomorphism $\Psi(\dot w) : \mathrm{pr}_1^*\mathcal{L}_I(\lambda)\to\mathrm{pr}_2^*\mathcal{L}_J(w^{-1}\lambda)$ (obtained in Proposition 2.1.2 with Remark 2.1.3) extends over the closure of $O_{IJ}(w)$ in $X_I\times X_J$ if and only if $\lambda\in D_I(w)$; and then it vanishes precisely on the closures of $O_{IJ}(w')$ for $w'\in\partial_I^\lambda(w)$.*

Proof. Let Z be the closure of $O_{IJ}(w)$. The normality of Schubert varieties ([1] and [12]) implies that Z is normal. Indeed let S be the closure of P_IwP_J/P_J in X_J and set $X = X_\emptyset$; we have that S is normal since it is the image by $\tau_J : X\to X_J$ of the Schubert variety in X associated to the longest element v in W_IwW_J (cf. [10], Ch. 8, Ex. 2.11); we have $Z = \phi_2(\phi_1^{-1}(S))$ with

$$X_I\times X_J \xleftarrow[(gP_Ig^{-1},x)\leftarrow\!\shortmid(g,x)]{\phi_2} G\times X_J \xrightarrow[(g,x)\mapsto g^{-1}x]{\phi_1} X_J, \tag{10}$$

whence Z is normal (cf. [10], Ch. 8, 2.25 and Ex. 2.11).

We choose a reduced expression $v = s_1\cdots s_\ell$ of the longest element v of W_IwW_J, as in the beginning of this subsection. Let Y be the closure of $O(v)$ in $X\times X$. Recall the Hansen-Demazure desingularization of Y. We inductively define $\overline{O}(s_1,\dots,s_i)$ for $1\le i\le\ell$ by

$$\overline{O}(s_1,\dots,s_i) = \overline{O}(s_1,\dots,s_{i-1})\times_X\overline{O(s_i)} \tag{11}$$

with $\overline{O}() = X$. We have an iterated fiber space

$$\overline{O}(s_1,\dots,s_\ell)\to\overline{O}(s_1,\dots,s_{\ell-1})\to\cdots\to\overline{O}(s_1)\to X. \tag{12}$$

Let D_i be the divisor on $\overline{O}(s_1,\dots,s_\ell)$ obtained as the inverse image of the divisor on $\overline{O}(s_1,\dots,s_i)$ defined by $O(1)\subset\overline{O(s_i)}$. Let f be the morphism $\overline{O}(s_1,\dots,s_\ell)\to Y$ sending $(B_0,\dots,B_\ell)$ to (B_0,B_ℓ). This is a desingularization of Y and the union D of D_i $(1\le i\le\ell)$ is a normal crossing divisor with $\overline{O}(s_1,\dots,s_\ell)\setminus D\simeq O(v)$.

We have $\tau_I^{-1}(O_{IJ}(w)) = \bigcup_{x\in W_IwW_J}O(x)$, whose closure is Y. In [5], (9.5) it was proved that the section $\tau_I^*\Psi(\dot w)$ extends to Y if and only if $\lambda\in D_\emptyset(v)$. The latter condition is equivalent to $\lambda\in D_I(w)$, since $\lambda\in\mathrm{X}(P_I)$. The former condition is equivalent to that $\Psi(\dot w)$ extends to the closure Z of $O_{IJ}(w)$, since τ_I is proper with connected fibers ($\simeq P_I/B$) and Z is normal. Moreover as in the proof of [5], (9.5) the order along D_i of $f^*\tau_I^*\Psi(\dot w)$ (considered as a rational function) is equal to $\langle\lambda,(v_i\alpha_i)^\vee\rangle$, where $v_i = s_1\cdots s_{i-1}$ as above. This implies that $\Psi(\dot w)$ vanishes only on the closures of $O_{IJ}(w')$ for $w'\in\partial_I^\lambda(v) = \partial_I^\lambda(w)$. □

3 The affineness of distinguished Deligne-Lusztig varieties

We state and prove our results on the (quasi-)affineness of the distinguished Deligne-Lusztig varieties.

3.1 The quasi-affineness

Before dealing with the affineness, we give a remark about the quasi-affineness. Although Theorem 3.1.1 below looks a part of the folkfore (cf. Remark 3.1.2), it would be meaningful to give a proof along the original approach by Haastert [7].

Let I be a subset of Δ. Set $\Pi_I = \Delta \setminus I$. We define the *chamber* in $\mathrm{X}(P_I)$ by

$$C_I^0 = \{\mu \in \mathrm{X}(P_I)_{\mathbb{R}} \mid \langle \mu, \alpha^\vee \rangle > 0 \text{ for } \alpha \in \Pi_I\}. \tag{13}$$

Let F^* be the endomorphism of $\mathrm{X}(T)$ sending x to the composition $x \circ F$. Let F_* be the endomorphism of $\mathrm{Y}(T)$ sending y to the composition $F \circ y$. By [15], (11.2), there exists a collection of powers $q(\alpha)$ $(\alpha \in \Phi)$ of p such that $F^*\sigma(\alpha) = q(\alpha)\alpha$ in $\mathrm{X}(T)$, see loc. cit. §11 for the details of $q(\alpha)$. We have $F_*(\alpha^\vee) = q(\alpha)(\sigma\alpha)^\vee$, since

$$\langle \sigma\beta, F_*(\alpha^\vee)\rangle = \langle F^*\sigma\beta, \alpha^\vee\rangle = q(\beta)\langle \beta, \alpha^\vee\rangle = q(\alpha)\langle \sigma\beta, (\sigma\alpha)^\vee\rangle \tag{14}$$

for any $\beta \in \Phi$, where the last equality is proven in the proof of [15], (11.5).

Let $w \in W$. Assume $I = w\sigma I$. Then the endomorphism F^*w^{-1} of $\mathrm{X}(T)$ induces an endomorphism of $\mathrm{X}(P_I)$, since for $\lambda \in \mathrm{X}(P_I)$ and for $\alpha \in I$ we have

$$\langle F^*w^{-1}\lambda, \alpha^\vee\rangle = \langle \lambda, wF_*(\alpha^\vee)\rangle = q(\alpha)\langle \lambda, (w\sigma\alpha)^\vee\rangle = 0. \tag{15}$$

Theorem 3.1.1. *All distinguished $X_I(w)$ are quasi-affine.*

Proof. Let L_w be the endomorphism on $\mathrm{X}(T)$ sending μ to $F^*w^{-1}\mu - \mu$. We first claim that L_w is injective. Consider the endomorphism ${}^wF$ of G sending x to $wF(x)w^{-1}$. Since some power of ${}^wF$ is a composition of the frobenius map and an inner automorphism, the kernel of ${}^wF$ is finite. Applying [15], (10.1) to ${}^wF$ on T, the endomorphism of T sending t to $t^{-1}\,{}^wF(t)$ is surjective. This implies that L_w on $\mathrm{X}(T)$ is injective. Then the endomorphism on $\mathrm{X}(P_I)$ obtained by restricting L_w to $\mathrm{X}(P_I)$ is also injective. Hence there exists $\lambda \in \mathrm{X}(P_I)$ such that $L_w(\lambda) \in -C_I^0$. This means that the invertible sheaf $\mathcal{L}$ associated to $L_w(\lambda)$ is ample. It follows from Proposition 2.1.2 that the restriction of $\Psi(\dot{w})$ to $X_I(w)$ gives a no-where vanishing section of $\mathcal{L}$ on $X_I(w)$, namely $\mathcal{L}$ is isomorphic to the structure sheaf over $X_I(w)$. Hence $X_I(w)$ is quasi-affine by [6], 5.1.2 on p. 94. □

Remark 3.1.2. Here is an alternative proof of this theorem, which the author learned from one of the referees. Consider

$$Y_I(w) := \{g \in G \mid g^{-1}Fg \in \dot{w}U_{\sigma I}\}/U_I \cap {}^{\dot{w}}U_{\sigma I},$$

which is a locally closed subvariety of G/U_I. Using $I = w\sigma I$, we have that $X_I(w)$ is the quotient of $Y_I(w)$ by the finite group consisting of ${}^wF$-fixed points on L_I. The theorem follows from the facts that G/U_I is quasi-affine ([13], Theorem 3) and that the quotient of a quasi-affine variety by a finite group is also quasi-affine ([13], Lemma 2 to Theorem 1 and [14], III, §3, 12).

3.2 A criterion for the affineness

Let $w \in W$. We assume $I = w\sigma I$. Let $\lambda \in D_I(w)$ and set

$$X_I^\lambda(w) := \overline{X_I(w)} - \bigcup_{v \in \partial_I^\lambda(w)} \overline{X_I(v)}. \tag{16}$$

Theorem 3.2.1. *Let J be the smallest subset of Δ containing I with $F^*w^{-1}\lambda - \lambda \in -C_J^0$. If the restriction to $X_I^\lambda(w)$ of $\tau_{IJ} : X_I \to X_J$ is quasi-finite, then $X_I^\lambda(w)$ is affine.*

Remark 3.2.2. (1) If there exists $\lambda \in D_I(w)$ such that $I = J$ (i.e., $F^*w^{-1}\lambda - \lambda \in -C_I^0$), then the quasi-finiteness condition is trivially satisfied. In this case $X_I^\lambda(w)$ is affine.

(2) Let $w' \in W_I \backslash W / W_{\sigma I}$ with $X_I(w') \subset X_I^\lambda(w)$ and set $\mathcal{C}(w') = P_I w' P_{\sigma I}$. That $\tau_{IJ}|_{X_I(w')}$ is quasi-finite is equivalent to that for every $x \in \mathcal{C}(w')$ there are only finitely many $h \in P_J/P_I$ such that $h^{-1}xFh \in \mathcal{C}(w')$. In particular every fiber of $\tau_{IJ}|_{X_I^\lambda(w)}$ consists of one point if and only if for every w' as above and for every $u \in W_J \setminus W_I$, we have $u^{-1}\mathcal{C}(w')Fu \cap \mathcal{C}(w') = \emptyset$.

(3) If $X_I^\lambda(w)$ is affine, then $X_I(w)$ is affine. In general, we claim that for any affine scheme X which is regular in codimension one and for any locally principal closed subscheme Y of X associated to a Cartier divisor D, we have that $X \backslash Y$ is affine. Since any invertible sheaf on an affine scheme is ample, we have an immersion $X \to \mathbf{P}^N$ associated to some power of $\mathcal{L}(D)$. Let $\overline{X}$ be the closure of the image. Then $X \setminus Y$ is isomorphic to the intersection of two affine subschemes X and $\overline{X} \setminus H$ for a hyperplane H in $\mathbf{P}^N$. The claim follows from the well-known fact that the intersection of open affine subschemes of a scheme separated over an affine scheme is affine.

Proof of Theorem 3.2.1. Put $Y = X_I^\lambda(w)$. Let Z be the closure of Y in X_I. Since we have $X_I^\lambda(w) = X_I^{n\lambda}(w)$ for any natural number n, we if necessary replace λ by $n\lambda$ so that λ belongs to $\mathrm{X}(P_I)$. Let $\mathcal{L}$ be the invertible sheaf associated to $F^*w^{-1}\lambda - \lambda$. Let V be the open subscheme of $\mathbf{P}(H^0(Z, \mathcal{L})^*)$ where $\Psi(\dot{w})$ does not vanish. By Proposition 2.2.1 we have a Cartesian product:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\phi} & \mathbf{P}(H^0(Z, \mathcal{L})^*). \end{array}$$

Note that V is affine and that ψ is proper (since ϕ is proper). Hence, in order to see that Y is affine, it suffices to show that ψ is quasi-finite. Let x be a point of the image of ψ. Let $y \in Y$ be any preimage of x. Let φ be a morphism $Y \hookrightarrow X_I \to \mathbf{P}(H^0(X_I, \mathcal{L})^*)$ determined by $\mathcal{L}$. Since $H^0(Z, \mathcal{L})$ contains every element obtained by restricting an element of $H^0(X_I, \mathcal{L})$, the point $x' := \varphi(y)$ is determined by x. This means $\psi^{-1}(x) \subset \varphi^{-1}(x')$. Since $F^*w^{-1}\lambda - \lambda \in -C_J^0$, the morphism φ factors as $Y \hookrightarrow X_I \to X_J \hookrightarrow \mathbf{P}(H^0(X_I, \mathcal{L})^*)$. Hence the quasi-finiteness of the restriction to $X_I^\lambda(w)$ of $\tau_{IJ} : X_I \to X_J$ shows that $\psi^{-1}(x)$ is finite. □

Lemma 3.2.3. *Let $\lambda \in \mathrm{X}(P_I)_{\mathbb{Q}}$. If $F^*w^{-1}\lambda \in -C_I^0$, then we have $\lambda \in D_I^0(w)$.*

Proof. Let $\alpha \in \Sigma_I^+(w)$ $(= \Sigma_I^+ \cap w\Sigma_{\sigma I}^-)$. Put $\beta := (w\sigma)^{-1}\alpha$. Since $\beta \in \Sigma_I^-$, we have $\langle \lambda, \alpha^\vee \rangle = q(\beta)^{-1} \langle F^*w^{-1}\lambda, \beta^\vee \rangle > 0$. □

We write $\Pi_I = \{\alpha_1, \ldots, \alpha_r\}$ so that $\Pi_I \cap w\Sigma_{\sigma I}^+ = \{\alpha_1, \ldots, \alpha_c\}$ and $\Pi_I \cap w\Sigma_{\sigma I}^- = \{\alpha_{c+1}, \ldots, \alpha_r\}$. Put $\beta_i = (w\sigma)^{-1}\alpha_i$. Note that $\beta_i \in \Sigma_I^+$ for $i \leq c$ and $\beta_i \in \Sigma_I^-$ for $i > c$. Let ω_i be the fundamental weight corresponding to α_i for $1 \leq i \leq r$.

Corollary 3.2.4. *If $q(\beta) > \sum_{i=1}^c \langle \omega_i, \beta \rangle$ for every $\beta \in \{\beta_1, \ldots, \beta_c\}$, then $X_I(w)$ is affine.*

Proof. Put $\mu = \sum_{j=1}^r \varepsilon_j \omega_j$ with positive rational numbers ε_j. Let λ be the element of $\mathrm{X}(P_I)_{\mathbb{Q}}$ with $\mu = -F^*w^{-1}\lambda$. By the above lemma we have $\lambda \in D_I^0(w)$. By Remark 3.2.2, (1), it suffices to show that $F^*w^{-1}\lambda - \lambda \in -C_I^0$ for some choice of ε_j $(j = 1, \ldots, r)$. We have

$$\langle F^*w^{-1}\lambda - \lambda, \alpha_i^\vee \rangle = \langle \mu, -\alpha_i^\vee + q(\beta_i)^{-1}\beta_i^\vee \rangle = -\varepsilon_i + q(\beta_i)^{-1} \sum_{j=1}^r \varepsilon_j \langle \omega_j, \beta_i^\vee \rangle. \tag{17}$$

For $i > c$, this is negative since $\beta_i \in \Sigma_I^-$. For $i \leq c$, the assumption $q(\beta_i) > \sum_{i=1}^{c} \langle \omega_i, \beta_i^\vee \rangle$ implies that $\langle F^* w^{-1}\lambda - \lambda, \alpha_i^\vee \rangle < 0$ for $\varepsilon_1 = \cdots = \varepsilon_c = 1$ and sufficiently small $\varepsilon_{c+1}, \ldots, \varepsilon_r$. □

What we can say for $|I| \geq |\Delta| - 2$ is as follows.

Corollary 3.2.5. (1) *If $|I| = |\Delta| - 1$, then $X_I(w)$ is affine.*

(2) *Assume $|I| = |\Delta| - 2$. We choose $\alpha_1, \alpha_2 \in \Pi_I$ so that $\beta_1 > \beta_2$ with $\beta_i = (w\sigma)^{-1}\alpha_i$. Assume that $\langle \omega_1, \beta_1^\vee \rangle < q(\beta_1)$ if $w \neq 1$. Then $X_I(w)$ is affine.*

Proof. (1) If $c = 0$, then this follows from Corollary 3.2.4. If $c = 1$, then w stabilizes Σ_I^+, whence $w = 1$; then $X_I(w)$ consists of finitely many points.

(2) If $c \leq 1$, then this follows from Corollary 3.2.4 and the assumption $\langle \omega_1, \beta_1^\vee \rangle < q(\beta_1)$. If $c = 2$, then w stabilizes Σ_I^+, whence $w = 1$; then $X_I(w)$ consists of finitely many points. □

Here is a remark to Corollary 3.2.5, (2).

Remark 3.2.6. Assume that α_1 and α_2 are in a σ-stable irreducible component of Φ, say Φ'. Let us look at the condition $\langle \omega_1, \beta_1^\vee \rangle < q(\beta_1)$ for each type of (Φ', σ). First assume that σ preserves lengths of roots (in this case $q(\alpha) = q$ for all $\alpha \in \Phi'$). The number $\langle \omega_1, \beta_1^\vee \rangle$ is at most the biggest number m of $\langle \omega, \beta^\vee \rangle$ where ω is a fundamental weight and β is a root with the same length as the simple root α corresponding to ω. This number m is 1 for A_n and B_2, and 2 for B_n, C_n and D_n ($n \geq 3$), and 3 for E_6, and 4 for E_7, and 6 for E_8, and 3 for F_4, and 2 for G_2. If σ does not preserve lengths of roots, the maximal possible number of $\langle \omega_1, \beta_1^\vee \rangle$ is 1 for ${}^2B_2$, and 4 for ${}^2F_4$, and 3 for ${}^2G_2$.

Corollary 3.2.7. *All distinguished Deligne-Lusztig varieties associated to rank-2 groups are affine.*

Proof. It remains to study the case of G_2 with $q = 2$ (and $I = \emptyset$) and the case of ${}^2G_2$ with $q = \sqrt{3}$ (and $I = \emptyset$), since to the other cases Corollary 3.2.5 is applicable by Remark 3.2.6. In a straightforward way, one can check that the w's to which Corollary 3.2.5 is not applicable are of length 3 for G_2 and of length 2 or 4 for ${}^2G_2$. In order to reduce cases, we choose $\alpha_1, \alpha_2 \in \Delta$ so that $\beta_1 > \beta_2$ with $\beta_i = (w\sigma)^{-1}\alpha_i$. Let s_1 and s_2 be the simple reflections corresponding to α_1 and α_2 respectively. The above w's are expressed as $s_2s_1s_2$ for G_2 and as s_2s_1 or $s_2s_1s_2s_1$ for ${}^2G_2$. (Two cases in each are treated simultaneously.)

Let $\lambda \in \mathrm{X}(P_I)_{\mathbb{Q}}$ with $\omega_1 = -F^*w^{-1}\lambda$. By Lemma 3.2.3, we have $\lambda \in D_I(w)$. Now we apply Theorem 3.2.1 to λ. We have $J = \{\alpha_1\}$ and $X^\lambda(w) = X(w) \cup X(w')$, where $w' = s_2s_1$ in the G_2-case, $w' = s_2$ in the ${}^2G_2$-case with $\ell(w) = 2$ and $w' = s_2s_1s_2$ in the ${}^2G_2$-case with $\ell(w) = 4$. It is clear from [4] §2, 1, (3) and (3′) on p.16 that $s_1^{-1}\mathcal{C}(v)Fs_1 \cap \mathcal{C}(v) = \emptyset$ for $v = w$ and w'. Hence it follows from Remark 3.2.2, (2) that every fiber of $\tau_{\emptyset J}|X^\lambda(w)$ consists of one element. Thus $X^\lambda(w)$ is affine by Theorem 3.2.1; hence $X(w)$ is affine, see Remark 3.2.2, (3). □

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